POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-7A : DIFFERENTIAL EQUATIONS, INTEGRAL TRANSFORMATIONS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(*Notations and symbols have their usual meanings.*)

Answer Question No. **1** and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

a) Show that the Fourier transform of a function, if exists, is bounded.

b) Using Fourier inversion formula, show that

$$\int_0^{\alpha} \frac{\cos \alpha x}{\alpha^2 + b^2} \, d\alpha = \frac{\pi}{2b} e^{-bx}, \; x > 0$$

c) Find the Fourier transform of the function —

$$f(x) = 1 - x^2, \text{ for } |x| \le 1$$

$$= 0, \quad \text{for } |x| > 1$$

d) Find the Laplace transform of the function —

$$f(t) = (1 - 2t)e^{-2t} .$$

e) Find $L^{-1}\left\{\dfrac{p+7}{p^2+2p+5}\right\}$, p being the Laplace transform parameter.

f) If $L\{f(t)\} = F(p)$, then find the value of $\lim_{p \to \infty} pF(p)$. (The necessary conditions may be assumed)

g) Define Hankel transform of order γ of a function $f(r)$, $0 \le r \le \infty$.

2. a) If the Fourier transform $F(k)$ of a function $f(x)$ exists, then prove that $F(k)$ is a continuous function of k.

   b) Find the Fourier transform of $f(x) = \frac{1}{x}$. 6 + 4

3. a) Using Parseval's relation for Fourier transforms, show that

   $$\int_0^\infty \frac{\sin ax \sin bx}{x^2} \mathrm{d}x = \frac{\pi a}{2}, \text{ where } 0 < a < b.$$

   b) If the Fourier sine transform of $f(x)$ is $\frac{k}{1+k^2}$, then find $f(x)$. 6 + 4

4. Solve the following heat conduction problem in an infinite rod :

   (i) $\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} + q(x, t), -\infty < x < \infty,$

   (ii) $u(x, 0) = 0, -\infty < x < \infty,$

   (iii) $u(x,t), \; u_x(x, t) \to 0$ as $|x| \to \infty$ 10

5. a) Find the solution of the following equation with the help of Laplace transform :

   $$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} + 5y = -8e^{-t}, \; y(0) = 2 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t} y(0) = 12.$$

   b) Find $L^{-1}\left\{ l_n \frac{p^2+1}{p(p+1)} \right\}$. 7 + 3

6. Let the function $F(p)$ be a given function of complex variable p in the domain $\mathrm{Re}(p) > a$ and is the Laplace transform of a function $f(t)$ of real variable t such that (i) $f(t) = 0$ for $t < 0$, (ii) in any finite interval of t, the function $f(t)$ is piecewise continuous and (iii) $f(t)$ is of exponential order $0(e^{at})$ *as* $t \to \infty$. Then prove that

   $$f(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{pt} \mathrm{d}p, \text{ where } r > a.$$ 10

7. a) Obtain the Laplace inversion of

$$F(p) = \frac{1}{\sqrt{1+p^2}}$$

in a series of powers of t.

Hence show that

$$\sin t = \int_0^t J_0(\tau) J_0(t-\tau)\, d\tau,$$

where $J_0(t)$ is Bessel's function of order zero.

b) Find the Hankel transform of —

$f(x) = 1,\ 0 < x < a,\ \ n = 0$

$= 0,\ x > a,\qquad n = 0$ 6 + 4