QP Code: 23/PT/13/VIA

POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-6A : GENERAL TOPOLOGY

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Prove that the following collection form a topology on a non-empty set : Fix a point $p \in E$.

$\tau_p = \{E\} \cup \{X \in P(E) : p \notin X\}$,

where $P(E)$ in the power set of E.

b) Let $\mathbb{R}$ be induced with discrete topology. Find the closure of the following subsets of $\mathbb{R}$.

(i) $\mathbb{Q}$, (ii) $\mathbb{R} - \mathbb{N}$ (iii) P = the set of all prime numbers

c) Give an example of a mapping which is continuous and closed but not open.

d) Show that the quotient topology on X is the largest topology such that $f : (Y, \tau) \to X$ is continuous, where (Y, τ) is a topological space.

e) In (X, τ) if for any pair of disjoint closed sets A and B there is a continuous function $f : X \to [0, 1]$ such that

$$f(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \in B \end{cases}.$$

Show that (X, τ) is normal.

f) If $\phi : (-1, 1) \to (\mathbb{R}, \tau)$, is given by

$\phi(x) = \dfrac{x}{1 - |x|}$, $-1 < x < 1$, examine if ϕ is a homeomorphism. Hence $(\mathbb{R}, \tau)$ is the usual topological space.

g) Examine if the real number space $\mathbb{R}$ with discrete topology is connected.

2. a) Show that a bijective continuous function from a compact space to a T_2-space is a closed map. 3

   b) Let $A=\{0\}\cup\left\{\frac{1}{n}| n\in \mathbb{N}\right\}$ be induced with usual topology τ_d. Show that every closed subset of A is compact (A, τ_d). 3

   c) Show that every second countable space (X, τ) is Lindelöff. 4

3. a) Show that a topological space (X, τ) is disconnected iff there in a continuous function $f:(X, \tau) \rightarrow \{0, 1\}$, which is onto. 4

   b) Let (X, τ_d) be a discrete topological space were $X=\{a, b, c, d\}$. Find all the components in X. 3

   c) Prove that the closure of a connected set in a topological space is connected. 3

4. a) Prove that a mapping $f:(X, \tau)\rightarrow(Y, \tau')$ is continuous iff $f:(\overline{A})\subset \overline{f(A)}$ for every subset A of X. 5

   b) Prove that a topological space (X, τ) is Hausdorff if every convergent net in X has a unique limit. 5

5. a) Show that a topological space is T_1 iff every singleton set is closed. 4

   b) Define a normal space. Show that a metric space is normal. 1 + 5

6. a) Define totally disconnected space. Show that the components of a totally disconnected space are its singletons. 1 + 4

   b) Define locally compact space. Show that every closed sub-space of a locally compact space is locally compact. 1 + 4

7. a) Define a uniform space (X, ν). If τ_ν is the uniform topology on X induced by ν, then show that the τ_ν - closure of $A=\overline{A}=\bigcap\{\cup(A):\cup\in\nu\}$. 1 + 4

   b) Show that a subnet of $\mathbb{R}$ with usual topology is connected iff it is an interval. 5