POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-3B : PARTIAL DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTION

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Define complete integral, general integral and singular integral of a first order partial differential equation $F(x, y, z, p, q) = 0$.

   b) If $z = f(x^2 - y) + g(x^2 + y)$, where f and g are arbitrary functions, prove that $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{1}{x}\dfrac{\partial z}{\partial x} = 4x^2 \dfrac{\partial^2 z}{\partial y^2}$.

   c) Formulate the partial differential equation by eliminating arbitrary constants c_1, c_2 from $(x - c_1)^2 + (y - c_2)^2 + z^2 = 1$.

   d) Obtain the complete integral of the equation

      $pqz = p^2(xq + p^2) + q^2(yp + q^2)$.

   e) If a harmonic function vanishes everywhere on the boundary, then prove that it is identically zero everywhere.

   f) Show that the equation

      $(1 - x^2)\dfrac{\partial^2 u}{\partial x^2} + 2y\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is hyperbolic for all (x, y) outside the region bounded by the circle $x^2 + y^2 = 1$, parabolic on the boundary of this region and elliptic for all (x, y) inside this region.

   g) Interpret the equations $Pp + Qq = R$ and $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ geometrically and establish the relationship between the two.

2. Reduce the equation

   $z_{xx} - 2\sin x\, z_{xy} - \cos^2 x\, z_{yy} - \cos x\, z_y = 0$ into its canonical form and hence solve it. 10

3. a) Show that a necessary and sufficient condition for the Pfaffian differential equation $\vec{X} \cdot d\vec{r} = 0$ to be integrable is that $\vec{X} \cdot \text{Curl}\, \vec{X} = 0$, where $\vec{X} = (P, Q, R)$ and $d\vec{r} = (dx, dy, dz)$.

   b) Check whether the differential equation

   $(y^2 + yz)\,dx + (xz + z^2)\,dy + (y^2 - xy)\,dz = 0$ is integrable or not. 8 + 2

4. a) Prove that the general solution of the linear partial differential equation $Pp + Qq = R$ is $F(u, v) = 0$, where F is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are solutions of the equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

   b) Solve : $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$. 6 + 4

5. Using method of separation of variables, solve the one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0, t) = u(l, t) = 0,\ t \ge 0$ and initial conditions $u(x, 0) = f(x)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x),\ 0 \le x \le l$. 10

6. Solve the one-dimensional diffusion equation $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$ in the region $0 \le x \le \pi,\ \ t \ge 0$ subject to the conditions

   (i) T remains finite as $t \to \infty$

   (ii) $T = 0$ if $x = 0$ and π for all t

   (iii) At $t = 0$, $T = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi \end{cases}$ 10

7. Using method of separation of variables, obtain the solution of the Laplace equation in two-dimension $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} = 0$ in a rectangle $0 \le x \le a,\ 0 \le y \le b$ subject to the boundary conditions $\psi(0, y) = \psi(a, y) = \psi(x, b) = 0$, $\psi(x, 0) = kx$, where k is a constant. 10