QP Code: 23/PT/13/IIIA

POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-3A : ORDINARY DIFFERENTIAL EQUATIONS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(*Symbols/notations have their usual meanings.*)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Show that $f(x, y) = xy^2$ satisfies Lipschitz condition on the rectangle $|x| \le 1, |y| \le 1$, but does not satisfy Lipschitz condition on the region $|x| \le 1, |y| < \infty$.

b) Verify whether the equation $\frac{dy}{dx} = 2y^{\frac{1}{2}}$, $y(0) = 0$ has unique solution or not.

c) Apply Picard's method to solve the given initial value problem up to third approximation : $\frac{dy}{dx} = 2y - 2x^2 - 3$ given that $y = 2$ when $x = 0$.

d) Solve $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 3y = 2\sec x$, $\sec x$ being a solution.

e) Determine the nature of the critical point for the following system : $\dot{x} = -4x - y$, $\dot{y} = x - 2y$.

f) Prove that $H_n(z) = (-1)^n e^{z^2} . \frac{d^n}{dz^n}(e^{-z^2})$ where $H_n(z)$ is the Hermite's polynomial of degree n.

g) Prove that $L''_n(0) = \frac{1}{2}n(n-1)$ where $L_n(z)$ is the Laguerre polynomial of degree n.

2. a) Solve and find the singular solution of

$$p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0, \; p = \frac{dy}{dx}.$$

b) Show that the Green's function for the equation $\frac{d^2u}{dx^2} - \alpha^2 u = f(x),\ 0 \le x \le 1$ subject to the boundary conditions $u(0) = 0,\ u(1) = 1$ is given by

$$G(x, \xi) = \begin{cases} \frac{\sin h\,\alpha(\xi - 1) . \sin h\,\alpha\, x}{\alpha . \sin h\,\alpha}, & 0 \le x \le \xi \\ -\frac{\sin h\,\alpha\,\xi\ .\ \sin h\,\alpha(x-1)}{\alpha\ .\ \sin h\,\alpha}, & \xi \le x \le 1 \end{cases}$$

Hence write the complete solution. 5 + 5

3. a) If $\vec{\phi}_1, \vec{\phi}_2, \ldots\ldots, \vec{\phi}_n$ be a fundamental set of solutions of the linear homogeneous vector differential equation $\frac{dx}{dt} = A(t)x$ and $\vec{\phi}$ be an arbitrary solution of that equation then prove that $\vec{\phi}$ can be expressed as a liner combination of $\vec{\phi}_1, \vec{\phi}_2, \ldots\ldots, \vec{\phi}_n$ for all $t \in [a, b]$.

b) Find the general solution of the homogeneous linear system $\frac{dx}{dt} = Ax$ where $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. 5 + 5

4. a) If u_1 and v_1 are different eigen functions to distinct eigen values λ_1 and λ_2 of a regular Sturm-Liouville equation

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + \{\lambda\rho(x) - q(x)\}\, u = 0,$$

then prove that eigen functions are orthogonal with weight function ρ where λ is a parameter, p, q are real valued functions of x, p and ρ being positive.

b) Find the eigen values and eigen functions of the differential equation $\frac{d^2y}{dx^2} + \lambda y = 0\ (\lambda > 0)$ satisfying the boundary conditions $y(0) = 0$ and $y(\pi/2) = 0$. 5 + 5

5. a) Classify the equilibrium points of $\dot{x} = (x - y)$, $\dot{y} = (x^2 - 1)$.

   b) Find the general solution of

   $$\sin^2 x \frac{d^2 y}{dx^2} - 2\sin x \cos x \frac{dy}{dx} + (1 + \cos^2 x) y = \sin^3 x$$ given that y = sin$x$ and $y = \sin x$ and $y = x \sin x$ are linearly independent solutions of the corresponding homogeneous equation. 5 + 5

6. a) Expand $z^3 + z^2 - 3z + 2$ in a series of Laguerre polynomials.

   b) If $n > -1$, show that $\int_0^z z^{n+1} J_n(z)\, dz = z^{n+1} . J_{n+1}(z)$ where $J_n(z)$ is the Bessel's function of the first kind of order *n*. 5 + 5

7. a) Prove that $J_n = \frac{1}{\pi} \int_0^\pi \cos(n\theta - z\sin\theta)\, d\theta$, n being an integer where $J_n(z)$ is the Bessel's function of the first kind of order *n*.

   b) If $m < n$, show that (i) $\int_{-1}^{1} z^m p_n(z) = 0$

   and (ii) $\int_{-1}^{1} z^n p_n(z) = \frac{2^{n+1} . (n!)^2}{(2n+1)!}$

   where $p_n(z)$ is the Legendre's polynomial of degree *n*. 5 + 5