QP Code: 23/PT/13/IIB

POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-2B : COMPLEX ANALYSIS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(*Symbols have their usual meanings.*)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) State Cauchy-Hadamard theorem for a power series.

   b) Evaluate $\int_C \frac{dz}{z-a}$ when 'a' is inside C and C is any simple closed contour.

   c) Show that a function which is analytic in the extended complex pane must be constant.

   d) Determine all bilinear transformations which have fixed points $-i$ and i.

   e) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{z^n}{2^n+1}$.

   f) Is the function $f(z) = \frac{1}{z}$ uniformly continuous in the region $|z| < 1$? Give reason.

   g) Find the residues of the function $f(z) = \frac{e^{iz}}{1+z^2}$ at its singularities.

2. a) State and prove Cauchy integral formula for a function of a complex variable.

   b) If $f(z) = \begin{cases} \dfrac{x^2 y^3 (xy^2 - i)}{x^3 + y^6}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

   then show that $\frac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector, but not as $z \to 0$ in any other manner. 6 + 4

3. a) Given that $f(z)$ is continuous in a simply connected domain D and $\int_C f(z)dz = 0$ for every closed curve C in D. Prove that $f(z)$ is analytic in D.

   b) Evaluate $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$ where C is the circle $|z| = 2$. 6 + 4

4. a) State and prove Rouche's theorem. Hence prove that every polynomial of degree n has n zeros in the complex plane.

   b) Discuss the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3\,4^n}$. 6 + 4

5. a) By the method of contour integration show that

   $$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \frac{\pi}{12}.$$

   b) Prove that the sum of a uniformly convergent series of continuous function is continuous. 6 + 4

6. a) Prove that $z = a$ is a zero of order p of a function f, if and only if, in some neighbourhood of $z = a$, f can be expressed as $f(z) = (z-a)^p\,\phi(z)$ where $\phi(z)$ is analytic at $z = a$ and $\phi(a) \neq 0$.

   b) Show that the transformation $w = \frac{5-4z}{4z-2}$ transforms the circle $|z| = 1$ into a circle of unit radius in the w-plane. Find the centre of that circle. 5 + 5

7. a) State and prove Cauchy residue theorem.

   b) Locate and name all the singularities of

   $$f(z) = \frac{z-2}{z^2}\sin\frac{1}{z-1}.$$

   c) Prove that the limit point of zeros of an analytic function is an essential singularity, unless the function is identically zero. 4 + 3 + 3