POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-2A : REAL ANALYSIS & METRIC SPACES

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Let $A=\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Show that Inf $A = 0$.

b) Let $A=\bigcup_{n=1}^{\infty}\left(n-\frac{1}{3^n}, n+\frac{1}{3^n} \right)$. Show that A is measureable and find $m\,(\,A\,)$.

c) Let $f\,(x)=\frac{1}{4}+\sin x$ in $\frac{\pi}{2}\le x \le \frac{3\pi}{2}$. Find f^{+} and f^{-}.

d) Verify that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ is a Fourier series.

e) Let $G=\left\{ \frac{1}{2^n}+\frac{1}{3^m} \,:\, n, m \in \mathbb{N} \right\}$. Find the derived set G' of G.

f) Show that an isometric image of a complete metric space is complete.

g) Show that every contraction in a metric space is a uniformly continuous function.

h) Show that $\{(x,y)\in \mathbb{R}^2 : x>0,\ y=\sin\frac{1}{x}\} \cup \{(0,y)\in \mathbb{R}^2 : \ -1\le y\le 1\}$ is a connected subset of $\mathbb{R}^2$.

2. a) Let $\overset{b}{\underset{a}{\mathrm{V}}}(f)<\infty,\ a<c<b$, show that

$$\overset{b}{\underset{a}{\mathrm{V}}}(f)=\overset{c}{\underset{a}{\mathrm{V}}}(f)+\overset{b}{\underset{c}{\mathrm{V}}}(f)$$ 4

b) If $F_1, F_2, \ldots, F_n$ are bounded closed sets that are mutually disjoints then show that $m\left(\bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{n} m(F_i)$. 6

3. a) Let O be an open interval containing E. Then show that $m^*(E) + m_*(O \setminus E) = m(O)$. 4

b) Show that every step function over [a, b] is measurable. 4

c) Let f be the characteristic function of the set of all irrationals in [0, 1]. Evaluate $L - \int_0^1 f\, dx$. 2

4. a) State and prove Lebesgue Dominated Convergence Theorem. 7

b) If f is continuous in [a, b], and $g(x) = 0$ at $x = a$ and $g(x) = 1$ when $x > a$, show that

$$R - S\int_a^b f\, dg = f(a)\left(g(b) - g(a)\right).$$ 3

5. a) Obtain the Fourier series for $\cos kx$ in $[-\pi, \pi]$ and deduce that $\pi \cot k\pi = \frac{1}{k} + \frac{2k}{k^2 - 1} + \frac{2k}{k^2 - 4} + \ldots$, where k is not an integer. 4 + 2

b) For any set G in (X, d), show that $Diam(G) = Diam(\overline{G})$. What can you conclude about G if $Diam(G) = 0$? 3 + 1

6. a) Show that any sub-space of a separable metric space is separable. 5

b) If A and B are two subsets in (X, d) and B is compact, show that $dist(A, B) = 0$ if and only if $\overline{A} \cap B \neq \phi$. 5

7. a) Prove that every sequentially compact metric space is totally bounded and complete. 5

b) Show that a subset of reals with usual metric is connected if and only if it is an interval. 5