POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-1B : LINEAR ALGEBRA

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Find the dimension of the vector space of all $n \times n$ real symmetric matrices over $\mathbb{R}$.

   b) Let $A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 1 & -2 & 3 \end{pmatrix}$. Find the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix with respect to the standard ordered basis of $\mathbb{R}^3$ is A.

   c) If α, β be two orthogonal vectors in a Euclidean space V, then prove that $\|\alpha + \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$.

   d) Use Cayley-Hamilton theorem to find A^{50} where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

   e) Show that the matrix $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ is not diagonalizable.

   f) What are the irreducible factors of the minimal polynomial of the matrix $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

   g) Is the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 4 \end{pmatrix}$ positive definite ?

2. a) Let V be a vector space over a field F. Let $T: V \to V$ be a linear transformation.

    (i) Define an eigen value and an eigenvector of T.

    (ii) Define the eigen space of T associated with an eigen value of T. Define geometric multiplicity of an eigen value of T.

    (iii) Prove that any one-dimensional T-invariant subspace of V is an eigen space of T.

   b) What is meant by a linear functional on a vector space ? Suppose f is a non-zero linear functional on an n-dimensional vector space V over a field F. What is the nullity of f? (2 + 2 + 3) + (1 + 2)

3. a) Suppose V is a finite dimensional vector space over a field F with $\dim V = n$. Let $T: V \to V$ be a linear transformation. Prove that any two matrices of T are similar.

   b) Suppose A is an $m \times n$ matrix over a field F. Then $L_A : F^n \to F^m$, defined by $L_A(X) = AX$, is a linear transformation. Prove that the nullity of L_A is the same as the dimension of the solution space of $AX = 0$. Hence prove that if the number of variables is more than the number of equations in $AX = 0$ then it has a non-trivial solution. 5 + 5

4. a) Extend { (2, 3, − 1), (1, − 2, − 4) } to an orthogonal basis of the Euclidean space $\mathbb{R}^3$ and then find an associated orthonormal basis.

   b) Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation defined by $T(x, y) = (-y, x)$. Prove that the only subspaces of $\mathbb{R}^2$ invariant under T are $\mathbb{R}^2$ and $\{0\}$. 5 + 5

5. a) Prove that any orthonormal set in an inner product space is linearly independent.

   b) Suppose $B = \{v_1, \ldots, v_n\}$ is an orthonormal basis of an inner product space V. Let T be a linear operator on V and $A = (a_{ij})_{n \times n}$ be the matrix of T with respect to B. Prove that $a_{ij} = \left\langle Tv_j, v_i \right\rangle$, for all i, j with $1 \le i, j \le n$.

   c) Can there exist a non-identity 3 × 3 real symmetric matrix A such that $A^3 = I_3$? Here I_3 is the 3 × 3 identity matrix. 3 + 3 + 4

6. a) Show that the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & 3 \end{pmatrix}$ is non-singular and express it as a product of elementary matrices.

b) Let m (t) be the minimal polynomial of an $n \times n$ matrix A. Prove that the characteristic polynomial of A divides $(m(t))^n$. 5 + 5

7. a) What is Sylvester law of inertia of real quadric forms ?

b) Diagonalize the matrix $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$ orthogonally.

c) Show that the quadratic form $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 - 4x_2x_3 - 2x_3x_1$ is positive semi-definite. 2 + 5 + 3