QP Code: 23/PT/13/IA

POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-1A : ABSTRACT ALGEBRA

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Let $f : G \to G'$ be a group homomorphism and $a \in G$ such that $O(a) = n$. If f is injective then show that $O(f(a)) = O(a)$.

   b) Is the group $(\mathbb{Q}, +)$ isomorphic to the group $(\mathbb{Q}', \cdot)$? Justify.

   c) Is the group $(\mathbb{Z} \times \mathbb{Z}, +)$ a cyclic group ? Justify.

   d) Show that every group of order 10 contains a normal subgroup of order 5.

   e) Let R be a commutative ring with identity 1 and A, B be two ideals of R such that $A + B = R$. Show that $AB = A \cap B$.

   f) Is the set of all non-units of $\mathbb{Z}[i]$ forms an ideal of $\mathbb{Z}[i]$? Justify.

   g) Are the fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(i)$ isomorphic ? Justify.

   h) Define a Euclidean domain.

2. a) Let G be a group and $Z(G)$ be the centre of G. If the quotient group $G/Z(G)$ is cyclic then show that G is a commutative group. Hence conclude that $|Z(G')| \le \frac{1}{4} |G'|$ for a non-commutative group G'. 3 + 2

   b) State the First Isomorphism theorem of groups. By using this theorem, show that any epimorphism from the group $(\mathbb{Z}, +)$ onto itself is an isomorphism. 2 + 3

3. a) Prove that every non-trivial finite p-group (p is prime) has a non-trivial centre. 5

   b) Define Inn(G), the inner automorphism of G. Show that $G/Z(G) \cong Inn(G)$. 2 + 3

4. a) Prove that every group is isomorphic to a permutation group. 5

   b) Let G be a group of order p^2 where p is a prime. Prove that G is commutative. Is the converse true ? Justify your answer. 3 + 2

5. a) Let R be a commutative ring with identity. Show that $\dfrac{R[x]}{\langle x \rangle} \cong R$.

   Hence conclude that $< x >$ is a prime ideal of $\mathbb{Z}[x]$. 3 + 2

   b) Let R be a commutative ring with identity. If P is a prime ideal of R then show that $P[x]$ is a prime ideal of $R[x]$. Does this result true for maximal ideal ? Justify. 3 + 2

6. a) Let F be a field. Prove that $F[x]$ is a Euclidean domain. 5

   b) Find the greatest common divisor of the following polynomials over $\mathbb{Q}$, the field of rational numbers :

   $x^2 + 1,\ x^6 + x^3 + x + 1$ 5

7. a) Define a prime field. Give two examples of prime fields. Show that any infinite prime field is isomorphic to $\mathbb{Q}$. 1 + 2 + 2

   b) Define Galois field. Let $GF(p^n)$ be a Galois field of order p^n. Show that the multiplicative group G^* of all non-zero elements in $GF(p^n)$ is cyclic. 1 + 4