PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of proper lay-out of the meterials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that it may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great part of these efforts is still experimental–in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these to admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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Master of Business Administration (MBA)
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Master of Business Administration (MBA)

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Module 1

Module 2

Unit 1 ❑ Overview of Operations Research

Structure

1.1 Historical Development of Operations Research

The science of Operations Research (OR) originated during the Second World War. During that time the Military Commands of U.K. and U.S.A. called upon scientists from several disciplines and organized them into teams. The mission of these teams was to formulate specific proposals and plans to aid the Military commands to arrive at decisions on optimal utilization of scarce military resources and efforts, and also to implement the decisions effectively. The new approach of the scientific study of military operations was termed as Operational Research in U.K., while in U.S.A. it was called Operations Research.

Following the end of the war, the success of the military teams attracted the attention of industrial managers who were seeking solutions to their complex executive type problems. During the year 1950, OR achieved recognition as a subject worthy of academic study in the universities. Since then the subject has been given more and more importance in the fields of Economics, Management, Public Administration, Behavioral Sciences, Social Work, Mathematics, Commerce and Engineering.

1.2 Nature and Meaning of OR

OR has been defined in various ways. Below are some definitions of OR which have changed according to the development of the subject.

1. OR is a scientific method of providing executive departments with a *quantitative basis* for *decision* regarding the operations under their control. **[Morse and Kimbal in 1946]**

2. OR is a scientific method of providing executives with an *analytical* and *objective* basis for decisions. **[P.M.S. Blackett in 1948]**

3. The term 'OR' has hitherto-fore been used to connate various attempts to study operations of war by scientific methods. From a more general point of view, OR can be considered to be an attempt to study those *operations of modern society which involved organizations of men or of men and machines.* **[P.M. Morse in 1948]**

4. OR is the application of *scientific methods, techniques* and *tools* to problems involving the *operations of systems* so as to provide these in control of the operations with *optimum solutions* to the problem. **[Churchman, Acoff, Arnoff in 1957]**

5. OR is the art of giving *bad answers* to problems to which otherwise *worse answers* are given. **[T.L. Saaty in 1958]**

6. OR is a management *activity* pursued in two complementary ways – one half by the free and bold *exercise of commonsense* untrammeled by any routine, and other half by the application of a repertoire of *well established precreated methods* and techniques. **[Jagjit Singh in 1968]**

7. OR is the *attack* of modern methods on *complex problems* arising in the *direction and management* to large systems of men, machines, materials, and money in industry, business and defence. The distinctive approach is to develop a *scientific model* of the system, incorporating measurements of factors such as chance and

risk with which to predict and *compare* the outcomes of alternative *decisions, strategies* or *controls*. The purpose is to help management to determine its policy and actions scientifically. **[Operations Research Quarterly in 1971]**

8. Operations Research is the art of winning war without actually fighting it.

9. OR is an applied decision theory. It uses any *scientific, mathematical or logical means* to attempt to cope with the problems that confront the executive when he tries to achieve a through going rationality in dealing with his decision problems. **[Miller and Starr]**

10. OR is a scientific approach to problem solving for executive management. **[H.M. Wagner]**

11. OR is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the scientific method of analysis. **[C. Kittel]**

12. OR is the systematic method oriented study of the basic structure, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision making. **[E.L. Arnoff and M.J. Netzorg]**

13. OR is the application of *scientific methods* to problems arising from operations involving *integrated* systems of *men, machines and materials*. It normally utilizes the knowledge and skill of an inter-disciplinary research team to provide the managers of such systems with optimum operating solutions. **[Fabrycky and Torgersen]**

14. OR is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man-machine systems and OR workers are actively engaged in applying this knowledge to practical problems in business, government, and society. **[OR Society of America]**

15. OR is the application of scientific method by inter-disciplinary teams to problems involving the controls of organized (man-machine) systems so as to provide solutions which *best serve the purpose of the organization as a whole*. **[Ackoff and Sasieni in 1968]**

16. OR utilizes the planned approach *(updated scientific method)* and an *inter-disciplinary* team in order to represent complex functional relationships as mathematical models for purpose of providing a *quantitative basis* for decision making and *uncovering new problems* for quantitative analysis. **[Thieanf and Klekamp in 1975]**

There are three main reasons for which most of the definitions are not satisfactory –

(i) OR is not a science like any well-defined physical, biological, social phenomena. Unlike scientists in well-known disciples of science, Operation Researchers do not claim to know or have theories about operations. OR is definitely not a scientific research into the control of operations. It is essentially a collection of mathematical techniques and tools, which in conjunction with a systematic approach are applied to solve practical decision problems of an economic or engineering nature. Thus it is very difficult to define OR precisely.

(ii) OR is inter-disciplinary in nature having application not only in military affairs and business but also in medicine, engineering, physics and so on. It makes use of experience and expertise of people from different disciplines to develop new methods and procedures. Most definitions do not include this important characteristic, i.e. inter-disciplinary approach of OR and hence are not satisfactory.

(iii) Most of the definitions have been offered at different times of development of OR and hence tend to emphasize only one aspect of OR.

1.3 Modelling in Operations Research

A model in OR is simplified representation of an operation or a process in which only the basic aspects or the most important features of a problem under investigation are considered. A model should clarify the decision alternatives, their anticipated effects, indicate the relevant data for analyzing the alternatives and lead to informative conclusions. In order words, a model is an instrument used to arrive at a well-structured view of reality.

1.4 Classification of OR Models

The word "model" has several meanings, all of which are relevant to OR. For instance a 'model' may act as a substitute for representing reality, such as a small-scale model aero plane; it may imply some sort of idealization, such as a model plan for employment scheme of the unemployed; it may be a mathematical equation representing the relationship between constants and variables, and so on. Basic OR models are as follows :

(a) **Iconic Model :** Iconic models are identical representation of the system, either in reduced or enlarged form. For example, a photograph, the model of an atom, a small-scale model is easy to conceive, very specific and concrete. However, it cannot be easily used to determine or predict effects of important changes in the actual system.

(b) **Analogue Models :** In these models one set of properties is used to represent another set of properties. For instance, graphs and maps in various colours are analogue models where the different colours represent different characteristics, like brown represents land, blue represents water, yellow represents production, etc. Demand curves, frequency curves (in Statistics) are also analogue models of the behaviour of events.

Analogue models are less specific and less concrete than iconic models, but they are easier to manipulate. They are generally more useful than the iconic models because of their vast capacity to represent the characteristics of the real system.

(c) **Symbolic (Mathematical) Models :** In these models a set of mathematical symbols are used to represent the components of the real system. The components are related together by means of a set of mathematical equations which describe the behaviour (or properties) of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model.

A symbolic model is generally the easiest to manipulate experimentally, and it is most general and abstract.

Other types of models which often arise are as follows :-

(a) **Combined analogue and mathematical models :** Sometimes analogue models are also expressed in terms of mathematical symbols. Such models belong to both analogue and mathematical models. For example, a simulation model (which is essentially a computer assisted experimentation on a mathematical structure of a real time structure in order to study the system under a set of assumptions) is of the analogue type but uses formulae. This type of model is commonly used by managers to "simulate" decisions, by studying the activity of the firm summarized in a scaled-down period.

(b) **Function Models :** Models may also be grouped according to the functions performed. For example, a function may serve to acquaint the analyst with such things as blueprint of layout, tables carrying data, a schedule indicating a sequence of operations (like computer programming).

(c) **Quantitative Models :** These models are used to measure the observations. A unit of measurement of length, volume, degree of temperature, etc. are quantitative models. Other examples of quantitative models are :

  (i) *transformation models* which help to convert a measurement of one scale to another (e.g. logarithmic tables, Centigrade vs. Fahrenheit conversion scale), and

  (ii) the *test models* which act as 'standards' against which measurements are compared (e.g. a specified standard in production control, business dealings, the quality of a medicine).

(d) **Qualitative models :** Qualitative models are those that can be classified by the subjective description e.g. "economic models" and "business models", which represent the gathering of all models pertaining to economic or business problems, respectively, are qualitative models.

1.5 Phases of OR

In discussing the phases of OR we shall mainly consider the mathematical models.

The procedure for an OR study generally involves the following major phases :

Phase I : Formulation of the problem

The first phase of OR requires formulation of the problem in an appropriate form. This should clearly state the problem's elements which include the controllable (decision) variables, the uncontrollable parameters that may affect the possible solutions, the restrictions or constraints on the variables and the objectives for defining a good and improved solution.

Phase II : Construction of the model

The second phase of the investigation is concerned with reformulation of the problem in a form convenient for analysis. It requires identification of both static and dynamic structural elements and construction of mathematical formulae to explain the interrelationship among the elements. The mathematical model should include the following three basic sets of elements :

(a) decision variables and parameters

(b) restrictions and constraints

(c) objective function.

Phase III : Derivation of the solution

The third phase deals with mathematical calculations to derive the solution to the model. Frequently, a solution of the model refers to a set of values of the decision variables which optimizes one of the objectives and gives permissible levels of performance on the other objectives.

Phase IV : Updating the model

This phase involves checking of the model's validity. A model will be said to be valid if it can provide a reliable prediction of the system's performance. A good practitioner of OR realizes that his model must have a longer life and consequently he updates the model time to time by taking into account the past, present and future specifications of the problem.

Phase V : Controlling the solution

This phase of the study establishes control over the solution by proper feedback on variables which change significantly. The solution goes out of control as soon as one or more of the controlled variables change. As the conditions are constantly changing, the same model and the same solution may not remain valid for a long time.

Phase VI : Implementing the findings

The final phase of the study deals with implementation of the results of the model. This phase is primarily executed through the co-operation of the OR experts and those who are responsible for managing and operating the system.

1.6 General Methods for Deriving the Solution

In general, there are three methods used for solving OR models. They are as follows :

(i) **Analytical Method :** Analytical method involves use of classical mathematics, such as differential calculus, finite differences, for finding solution to the model. The kind of mathematics used depends on the nature of the model.

(ii) **Iterative Method :** When the analytical method fails to derive the solution the iterative method is used. Such a procedure starts with a trial solution and a set of rules for improving it. The trial solution is then replaced by an improved solution and the process is repeated till either no further improvement is possible or the cost of further computation is not justified.

(iii) **Simulation Method :** This method involves the use of probability and sampling concepts to estimate the values of the parameters involved in the model.

1.7 Common Problems of OR

Some of the commonly accepted well defined problems of OR can be classified as follows :

(a) **Allocation Problems :** This involves the optimum allocation of available resources so as to maximize profit or minimize cost subject to prescribed restrictions.

(b) **Competitive Problem :** Here one has to determine the strategies to be adopted by decision makers under competition or conflict.

(c) **Inventory Problem :** Such a problem requires determination of optimum (economic) order quantity and ordering (production) intervals so as to maximize the profit or minimize the cost involved.

(d) **Waiting Line Problem :** Here the problem is basically to organize service facilities so as to minimize the total cost of providing service and obtaining service, which are primarily related with the value of time spent by a customer in the queue.

(e) **Sequencing Problem :** A sequencing problem deals with scheduling of jobs through machines in such a way so as to minimize the total elapsed time.

(f) **Routing Problem :** Such a problem requires finding the optimal route to be taken in order to minimize the total cost or total time of traveling. One such problem is the traveling salesman problem.

(g) **Replacement Problem :** Replacement problems arise when the efficiency of components of a system under consideration decrease with time resulting in failure or break-down of the system. The problem is to decide when to replace a component in order that the total cost involved is minimized.

The tools used to solve the above problems are referred to as *techniques of OR.*

1.8 Salient Features of OR

The salient features of OR are as follows :

(a) It is an inter-disciplinary team approach for finding the optimum return.

(b) It uses techniques of scientific research to arrive at the optimum solution.

(c) It emphasizes on the over all approach to the system, i.e. all the aspects of the problem under consideration.

(d) It tries to optimize the total output so as to maximize the profit and minimize the loss or cost.

(e) It cannot give perfect answers to a problem, but can only improve the quality of the solution.

1.9 OR in India

In India Operations Research came into picture in 1949 when an OR unit was set up at the Regional Research Laboratory, Hyderabad. At the same time Prof. R.S. Verma (Delhi University) set up an OR team in Defense Science Laboratory to solve the problems of store purchase and planning. Prof. P.C. Mahalanobis established an OR team in Indian Statistical Institute, Kolkata in 1953 for solving the problem of national planning and survey. In 1957 the OR Society of India was formed and it became a member of the International Federation of OR Societies in 1960. It started a journal called OPSEARCH in 1963. Presently a number of OR journals are published in India viz. *Industrial Engineering and Management, Materials Management Journal of India, Defence Science Journal, SCIMA, Journal of Engineering Production,* etc.

The first important application of OR in India was made by Prof. P.C. Mahalanobis. He used the OR techniques to formulate the second five-year plan. Planning Commission employed OR methods for planning the optimum size of the Caravelle fleet of Indian Airlines. Some of the industries, viz. Hindustan Lever Ltd., Union Carbide, TELCO, Hindustan Steel, Imperial Chemical Industries, Tata Iron and Steel Company, Sarabhai Group, etc. have engaged OR teams. Kirloskar Company is using the assignment technique of OR to maximize profit.

As far as teaching of OR in India is concerned, the University of Delhi was the first to introduce a complete M.Sc. course in OR in 1963. Simultaneously, Institute of Management at Kolkata and Amedabad starting teaching OR in their MBA courses. OR is now being taught in almost all Institutes and Universities in various disciplines like Mathematics, Statistics, Commerce, Economics, Engineering, etc. Government has also introduced OR as a subject for the IAS, CA, ICWA, etc. examinations.

1.10 Questions

1. Define Operations Research.

2. Discuss the origin and development of Operations Research.
3. What is a model? Discuss the different types do models in Operations Research.
4. What are the different phases of model building?
5. Discuss the scope of Operations Research.

Unit 2 ❑ Linear Programming

Structure

2.1 Introduction

In 1947, George Dantzig and his associates, while working in the U.S. Department of Air Force, observed that a large number of military planning problems would be formulated as maximizing/minimizing a linear form of profit/cost function, linear in a number of variables which are restricted in values satisfying a set of linear constraints (equations or inequalities). Such a formulation of an optimization (maximization/minimization) problem is referred to as a Linear Programming Problem (LPP). The term 'Programming' refers to the process of determining a particular programme or course of action. Linear programming is one of the most important optimization techniques developed in the field of OR. Generally, a LPP can be written as to optimize (maximize) $z = c_1x_1 + c_2x_2 + ... + c_nx_n$ ... (2.1) subject to

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \le b_i,\ i = 1, 2, \dots, m$$
$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \le b_i,\ i = m_1 + 1, \dots, m_1 + m_2 \qquad \dots (2.2)$$
$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n > b_i,\ i = m_1 + m_2 + 1, \dots, m$$
$$x_j \ge 0, j = 1, 2, \dots, n \qquad \dots (2.3)$$

The problem is to find $x_1, \dots, x_n$, called decision variables, such that z is minimized. (2.1) is called the objection function of the problem and the linear restrictions (2.2) are known as constraints. The (2.3) are called non-negativity restrictions.

2.2 Formulation of LPP

Given an optimization problem, the primary task is to formulate it in an appropriate form. The outcomes of formulation of LP problems are explained through the following examples :

Example 1. (Production Allocation Problem). *A firm manufactures two type of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires one minute of processing time on G and two minutes on H; type B requires one minute on G and one minute on H. The machine G is available for not more than 6 hour 40 minutes while machine H is available for 10 hours during any working day.*

Formulate the problem as a linear programming problem.

Formulation. Let x_1 be the number of products of type A and x_2 the number of products of type B.

After carefully understanding the problem the given information can be systematically arranged in the form of the following table.

Table 2.1

Machine	Time of products (minutes)		Available time
	Type A (x_1 units)	Type B (x_2 units)	(minutes)
G	1	1	400
H	2	1	600
Profit per unit	Rs. 2	Rs. 3	

Since the profit on type A is Rs. 2 per product, $2x_1$ will be the profit on selling x_1 units of type A. Similarly, $3x_2$ will be the profit on selling x_2 units of type B. Therefore, total profit on selling x_1 units of A and x_2 unit of B is given by

$$P = 2x_1 + 3x_2 \text{ (Objective function)}$$

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required on machine G is given by : $x_1 + x_2$.

Similarly, the total number of minutes required on machine H is given by $2x_1 + x_2$.

But, machine G is not available for more than 6 hour 40 minutes (= 400 minutes). Therefore,

$$x_1 + x_2 \leq 400 \text{ (first constraint)}$$

Also, the machine H is available for 10 hours only, therefore,

$$2x_1 + x_2 \leq 600 \text{ (second constraint)}$$

Since it is not possible to produce negative quantities,

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ (non-negative restrictions).}$$

Hence allocation problem of the firm can be finally put in the form :

Find x_1 and x_2 such that the profit $P = 2x_1 + 3x_2$ is maximum, subject to the conditions :

$$x_1 + x_2 \leq 400,\ 2x_1 + x_2 \leq 600,\ x_1 \geq 0,\ x_2 \geq 0$$

Example 2. *A company produces two types of Hats. Each hat of the first type requires twice as much labour time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales the first and second type to 150 and 250 hats. Assuming that type profits per hat are Rs. 8 for type A and Rs. 5 for type B. formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.*

Formulation. Let the company produce x_1 hats of type A and x_2 hats of type B each day. So the profit P after selling these two products is given by the linear function:

$$P = 8x_1 + 5x_2 \text{ (Objective function)}$$

Since the company can produce at the most 500 hats in a day and A type of hats require twice as much time as that of type B, production restriction is given by $2tx_1 + tx_2 \leq 500t$, where t is the labour time per unit of second type, i.e.

$$2x_1 + x_2 \leq 500$$

But, there are limitations on the sale of hats, therefore further restrictions are :

$$x_1 \leq 150, x_2 \leq 250.$$

Also since the company cannot produce negative quantities,

$$x_1 \geq 0, x_2 \geq 0.$$

Hence the problem can be written as :

Find x_1 and x_2 such that the profit

$$P = 8x_1 + 5x_2 \text{ is maximum,}$$

subject to the restrictions

$$2x_1 + x_2 \leq 500, x_1 \leq 150, x_2 \leq 250$$
$$x_1 \geq 0, x_2 \geq 0,$$

2.3 Some Important Definitions

1. **Solution :** Any vector $x = (x_1, ..., x_n)'$ of variables satisfying constraints (2.2) is called a solution to the linear programming problem.

2. **Feasible solution :** Any solution to the linear programming problem is said to be feasible if it satisfies the non-negativity restrictions (2.3).

3. **Optimal solution :** Any feasible solution to the linear programming problem which optimizes the objective function (1) is called an optimal solution to the problem.

4. **Unbounded solution :** A feasible solution of the linear programming problem for which Z can be made infinitely large of small is called an unbounded solution.

2.4 Graphical Solution of LP Problem

A LP problem with only two decision variables can be easily solved by the graphical method. In the graphical method we first draw on graph, the straight lines defined by the equations corresponding to the inequality constraints. For example, if a constraint is $2x - 3y \le 5$, we draw the straight line given by the equation $2x - 3y = 5$. Next, for each constraint the region of points satisfying the constraint is shaded. The common shaded region contains points called feasible solution which simultaneously satisfy all the constraints. This region is called the feasible region or the region of feasible solution. Choosing an initial value of z the straight line corresponding to the objection function is drawn. This line is shifted parallel to itself over the feasible region in the direction of decreasing (increasing) value of z till it just but moves out of the region. The value of z corresponding to this position of the straight line gives the minimum (maximum) value of the objective function and all points on the portion of the line included in the feasible region given the optimal solutions.

Examples :

(1) **Problem having unique solution :**

Consider the problem

$$\begin{aligned} &\text{minimize } z = 1.5x_1 + 2.5x_2 \\ &\text{subject to } \; x_1 + 3x_2 \ge 3 \\ &\qquad\qquad\quad x_1 + x_2 \ge 2 \\ &\qquad\qquad\quad x_1, x_2 \ge 0. \end{aligned}$$

Graphical solution : The geometrical interpretation of the problem is given in Figure 2.1.

The minimum value of z is attained at the point of intersection of the straight lines $x_1 + 3x_2 = 3$ and $x_1 + x_2 = 2$. This point is (1.5, 0.5) and min $z = 3.5$. Here we have a unique optimal solution to the problem.

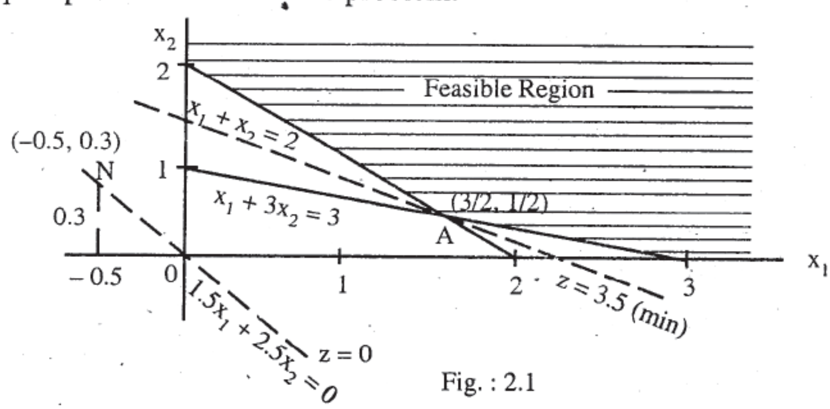


Fig. : 2.1

(2) **Problem having unbounded solution :**

Consider the problem

maximize $z = 3x_1 + 2x_2$

subject to $x_1 - x_2 \leq 1$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0.$

Graphical solution : The region of feasible solution is the shaded area in Figure 2.2. Clearly in this case z can be made arbitrarily large and so the problem has no finite maximum value of z. The optimal solution is in this case is unbounded.

(3) **Problem with more than one optimal solution.**

Consider the problem

maximize $z = -x_1 + 2x_2$

subject to $-x_1 + x_2 \leq 1$

$-x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0.$

Graphical solution : The graphical interpretation of the problem is given in Figure 2.3.

Here the straight line representing the objection function for $z = 4$ coincides with are edge of the feasible region. Thus, every point (x_1, x_2) lying on this edge $(-x_1 + 2x_2 = 4)$, which goes to infinity on the right, gives $z = 4$, which is the maximum value of z, and therefore every point on this edge is optimal solution.

(4) **Problem with inconsistent system of constraints.**

Consider the problem

maximize $z = 3x_1 - 2x_2$

subject to $x_1 + x_2 \leq 1$

$2x_1 + 2x_2 \geq 4$

$x_1, x_2 \geq 0.$

Graphical solution : The graphical representation of the problem is given in Figure 2.4. The figure shows that there is no point (x_1, x_2) which satisfies both the constraint simultaneously. Hence the problem has no solution as the constraints are inconsistent.

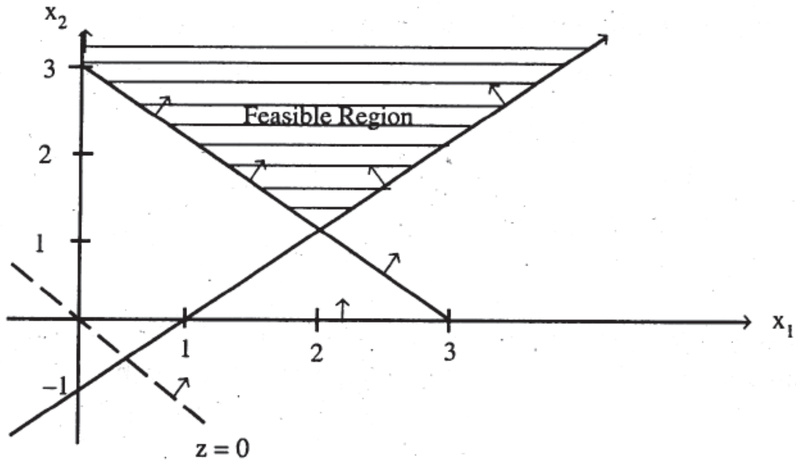


Fig. : 2.2

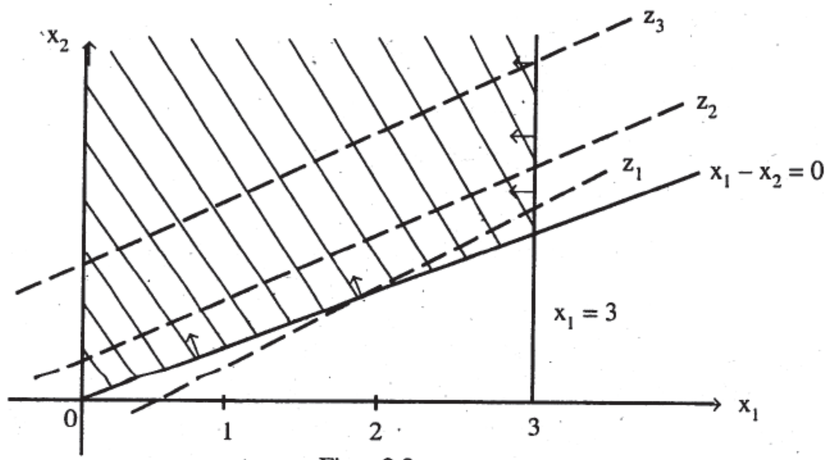


Fig. : 2.3

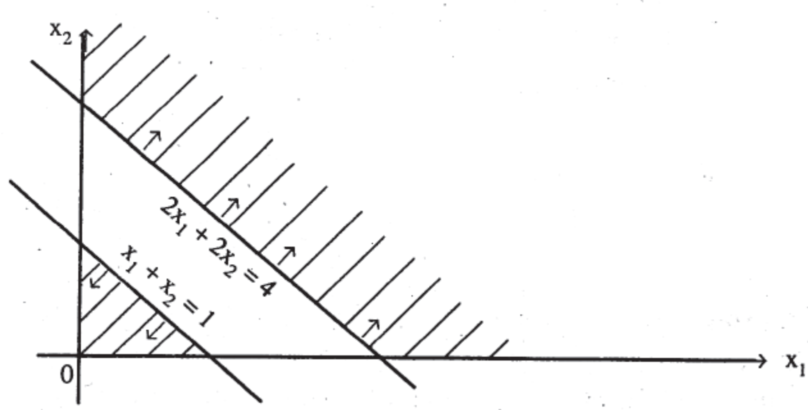


Fig. : 2.4

2.5 Standard Form of LP Problem

The standard form is used to develop procedures for solving a given LP problem. The standard form of an LP problem is as follows :

minimize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i, \; i = 1, 2, \ldots, m$.

$x_j \geq 0, j = 1, 2, \ldots, n.$

In matrix notation the above can be written as

minimize $z = c'x$ ... (2.4)

subject to $Ax = b$ ... (2.5)

$x \geq 0,$ ... (2.6)

where $c = (c_1\ c_2\ c_3 \ldots c_n)'$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$b = (b_1\ b_2 \ldots b_m)'$, $x = (x_1\ x_2 \ldots x_n)'$.

c is called the cost vector, X is the decision vector, b is the requirement vector and A is known as the coefficient matrix. In order that the constraint equations be linearly independent it is assumed that rank (A) = $m(\leq n)$. Also we take $b_i \geq 0, i = 1, 2, \ldots, m$.

2.6 Conversion of a given LP Problem to the Standard Form

Step 1 : If the given LP problem is a maximization problem, i.e. the problem is to maximize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ then it is reduced to a minimization problem using the fact that for any function $f(x)$,

$\max f(x) = -\min\{-f(x)\}$

For example, the problem

maximize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

is equivalent to

minimize $(-z) = -c_1x_1 - c_2x_2 + \ldots - c_nx_n$

i.e. minimize $z^* = c^*_1x_1 + c^*_2x_2 + \ldots + c^*_nx_n$

where $z^* = -z$

$c^*_j = -c_j,\ j = 1, 2, \ldots, n.$

Step 2 : If a constraint (other than the non-negativity constraints) is an inequality constraint, we convert it to an equality constraint by adding or subtracting a non-negative variable from the left hand side as the situation may be

For example, consider the constraints

$$x_1 + x_2 \leq 2$$
$$2x_1 + 4x_2 \geq 5.$$

We add a variable $x_3 = 2 - (x_1 + x_2) \geq 0$ to the l.h.s. of the first constraint to get

$$x_1 + x_2 + x_3 = 2.$$

From the l.h.s. of the second constraint we subtract $x_4 = 2x_1 + 4x_2 - 5 \geq 0$ to get the equality constraint

$$2x_1 + 4x_2 - x_4 = 5$$

Here x_3 is called a slack variable and x_4 is called a surplus variable and both the variables are non-negative.

Step 3 : If a decision variable x is unrestricted in sign (i.e. may be positive, negative or zero), it is replaced by two non-negative variables x' and x'' through the relation $x = x' + x''$, $x', x'' \geq 0$.

Note : It should be remembered that the coefficient of a slack or surplus variable in the objective function is always zero so that the conversion of inequality constraints to equations does not change the objective function.

Step 4 : If in any constraint the r.h.s. is negative, the constraint is multiplied by (– 1) to make the r.h.s. non-negative. For example, consider the constraint

$$3x_1 - 2x_2 \geq -5.$$

Multiplying by (– 1) the constraint becomes to

$$-3x_1 - 2x_2 \leq 5.$$

Addition of slack variable $x_3(\geq 0)$ gives the constraint as

$$-3x_1 - 2x_2 + x_3 = 5.$$

Example : Express the following LP problem in the standard form –

$$\begin{aligned} \text{maximize } z = \quad & -x_1 + 2x_2 - x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + 4x_3 \geq -4 \\ & 3x_1 + 5x_2 + 2x_3 \geq 7 \\ & x_1 \geq 0, x_2 \geq 0 \text{ unrestricted in sign.} \end{aligned}$$

Solution : Proceeding as in the steps above we get the problem in the standard form as follows :

$$\begin{aligned} \text{minimize } z^* = {} & x_1 - 2x_2 + (x'_3 - x''_3) \\ \text{subject to } \; & -2x_1 - 3x_2 - 4(x'_3 - x''_3) + x_4 = 4 \\ & 3x_1 + 5x_2 + 2(x'_3 - x''_3) - x_5 = 7 \\ & x_1 \geq 0, x_2 \geq 0, x'_3 \geq 0, x''_3 \geq 0, x_5 \geq 0. \end{aligned}$$

Here $z^* = -z$, $x_3 = x'_3 - x''_3$, and x_4 and x_5 are respectively the slack and surplus variables.

2.7 Simplex Method

When the number of decision variables is more than 2 it is not possible to solve the LP problem graphically. The simplex method provides an algorithm for solving a LP problem in such a case. For application of this method an LP problem has to be reduced to the standard form.

To understand the simplex method the following definitions and theorems are essential.

Definitions

1. **Basic solution (B. s.) :** A basic solution to the set of constraints (2.5) of the LP problem in the standard form (2.4) — (2.6) is a solution obtaining by setting any $(n - m)$ variables among $x_1, x_2, \ldots, x_n$ to zero and solving for the remaining m variables, provided the determinant of the coefficients of these m variables in (A) is non-zero. These m variables are called basic variables and the remaining $(n - m)$ variables, which are set to zero, are called non-basic variables.

Let B be a $n \times m$ non-singular submatrix of **A** formed by the coefficients of m variables denoted by the $m \times 1$ vector $\boldsymbol{x}_B$. Then setting the remaining $(n-m)$ variables to zero, from $\boldsymbol{Ax} = \boldsymbol{b}$ we get $\boldsymbol{Bx}_B = \boldsymbol{b}$, so that $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$. Then, $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_n \\ \boldsymbol{0} \end{pmatrix}$ is a basic solution. We often refer to x_B as the basic solution. Under the assumption of rank (**A**) = $\boldsymbol{m}$, we get $\boldsymbol{B}$ as a basis matrix of $\boldsymbol{A}$.

2. **Basic feasible solution (B.f.s.) :** A basic feasible solution is a basic solution satisfying the non-negativity constraints (2.6). Basic feasible solutions may be of two types—

   (a) **Non-degenerate b.f.s. :** A non-degenerate basic feasible solution is a basic feasible solution in which the basic variables are positive.

   (b) **Degenerate b.f.s. :** A degenerate basic feasible solution is a basic feasible solution with at least one basic variable zero.

3. **Optimum B.f.s. :** A basic feasible solution is said to be optimum if it minimizes the objective function (2.4).

1. **Hyperplane :** The set of points x in R^n (n-dimensional real space) satisfying $c_1x_1 + c_2x_2 + \ldots, c_nx_n = z$, or $c'x = z$ for prescribed values of $c_1, c_2, \ldots, c_n$ and z defines a hyperplane.

2. A hyperplane $\{x \varepsilon R^n \mid c'x = z\}$ divides R^n into three mutually exclusive sets, viz. $S_1 - \{x\varepsilon R^n \mid c'x < z\}$ $S_2 - \{x\varepsilon R^n \mid c'x - z\}$ and $S_3 - \{x\varepsilon R^n \mid c'x > z\}$

Open half spaces : The sets $S_1 = \{x\varepsilon R^n \mid c'x < z\}$ and $S_3 = \{x\varepsilon R^n \mid c'x > z\}$ are called open half spaces.

Closed half-spaces : The sets $S_4 - \{x\varepsilon R^n \mid c'x \leq z\}$ and $S_5 - \{x\varepsilon R^n \mid c'x \geq z\}$ are called half-spaces.

Convex sets : A set C in R^n is said to be a convex set if for any two distinct points $x_{(1)}, x_{(2)} \varepsilon C$, every point $\boldsymbol{x} = \lambda \boldsymbol{x}_{(1)} + (1-\lambda)\boldsymbol{x}_{(2)}$, $0 \leq \lambda \leq 1$, must also be in C.

Convex combination of points : A convex combination of points $x_{(1)}, x_{(2)}, \ldots, x_{(k)}$ in R^n is defined as a point

$$x = \lambda_1 x_{(1)} + \lambda_2 x_{(2)} + \ldots + \lambda_k x_{(k)},$$

where $\quad 0 \leq \lambda_i \leq i,\ i = 1, 2, \ldots, k,\ \sum_{i=1}^{k} \lambda_i = 1.$

Convex hull : The convex hull of any given set of points S is the set of all convex combinations of sets of points from S.

Convex polyhedron : If a set S contains a finite number of points, the convex hull of S is called a convex polyhedron.

Extreme points of a convex set : A point x is said to be an extreme point of a convex set C if it cannot be expressed as a convex combination of two distinct points in C.

Some important theorems

Theorem 1. A hyperplane in R^n is a convex set.

Proof. Let $S = \{x \varepsilon R^n \mid c'x = z\}$ be a hyperplane and also let x_1 and x_2 be any two points on the hyperplane. Then,

$$c'x_1 = z \text{ and } c'x_2 = z.$$

Therefore, for $0 \le \lambda \le 1$,

$$c'[\lambda x_1 + (1-\lambda)x_2] = c'(\lambda x_1) + [(1-\lambda)x_2] = c'(\lambda x_1) + (1-\lambda)c'x_2$$
$$= \lambda z + (1-\lambda)z = z.$$

Hence the point $\lambda x_1 + (1-\lambda)x_2$, for $0 \le \lambda \le 1$ lies in the hyperplane. So the hyperplane is convex.

Theorem 2. The closed half space $H_1 = \{x \mid c'x \ge z\}$ and $H_2 = \{x \mid c'x \le z\}$ are convex set.

Proof. Let $x^{(1)}$ and $x^{(2)}$ be any two points of H_1. Therefore,

$$c'x^{(1)}A \ge z \text{ and } c'x^{(2)} \ge z.$$

If $0 \le \lambda \le 1$, then

$$c'[\lambda x^{(1)} + (1-\lambda)x^{(2)}] = \lambda(c'x^{(1)} + (1-\lambda)c'x(2) > \lambda z + (1-\lambda)z = z]$$

Hence $x^{(1)}, x^{(2)} \varepsilon H_1$ and $0 \le \lambda \le 1 \Rightarrow [\lambda x^{(1)} + (1-\lambda)x^{(2)}] \varepsilon H_1$. So H_1 is convex.

Similarly, if $x^{(1)}, x^{(2)} \varepsilon H_2$, $0 \le l \le 1$, then replacing the inequality sign '$\le$' in above, it is true that

$$[\lambda x^{(1)} + (1-\lambda)x^{(2)}] \varepsilon H_2.$$

So H_2 is also convex.

Corollary. The open half spaces : $\{x \mid c'x > z\}$ and $\{x \mid c'x < z\}$ are convex sets.

Theorem 3. (a) The intersection of two convex sets is also a convex set.

(b) Interaction of any finite number of convex sets is also a convex set.

Proof. (a) Let C_1 and C_2 be two convex sets and also let $C = C_1 \cap C_2$.

To show that C is convex.

Let $x^{(1)}, x^{(2)} \varepsilon C$ and $S = \{x \mid x = \lambda x^{(1)} + (1-\lambda)x^{(2)}, 0 \le \lambda \le 1\}$

Now $\quad x^{(1)}, x^{(2)} \varepsilon C \Rightarrow x^{(1)}, x^{(2)} \varepsilon C_1 \quad$ (C_1 being convex)

Also, $\quad x^{(1)}, x^{(2)} \varepsilon C \Rightarrow x^{(1)}, x^{(2)} \varepsilon C_2 \quad$ (C_2 being convex)

Therefore, $x^{(1)}, x^{(2)} \varepsilon C \Rightarrow S \subset C_1$ and $S \subset C_2 \Rightarrow S \subset C_1 \cap C_2 \Rightarrow S \subset C$.

Hence C is convex.

(b) Let $C_1, C_2, ..., C_n$ be n convex sets and $C = C_1 \cap C_2 \cap, ..., \cap C_n$.

Now, $\quad x_1 \varepsilon C \Rightarrow x_1 \varepsilon C_i$, for all $i = 1, 2, ..., n$

and $\quad x_2 \varepsilon C \Rightarrow x_2 \varepsilon C_i$, for all $i = 1, 2, ..., n$.

Since C_i is convex set for all $i = 1, 2, ..., n$.

$$x_1, x_2 \varepsilon C \quad \Rightarrow \lambda x_1 + (1-\lambda)x_2 \varepsilon C_i, \text{ for all } i = 1, 2, ..., n, \text{ where } 0 \le \lambda \le 1$$

$$\Rightarrow \lambda x_1 + (1-\lambda)x_2 \subseteq (C_1, \cap C_2 \cap, ..., \cap C_n, 0 \le \lambda \le 1.$$

That is, $x_1 \varepsilon C$, $x_2 \varepsilon C$

$$\Rightarrow \lambda x_1 + (1-\lambda)x_2 \subseteq C; 0 \le \lambda \le 1,$$

Hence by definition $C_1, \cap C_2 \cap, ..., \cap C_n$ is a convex set.

Corollary. If C_1 and C_2 are closed convex sets, then C_1, C_2 is also a closed convex set.

Example : Show that $S = \{(x_1, x_2, ..., x_3) : 2x_1 - x_2 + x_3 \le 4, x_1 + 2x_2 - 2x_2 - x_3 \le 1\}$ is a convex set.

Solution : Obviously, S is the intersection of two half spaces, viz.

$$H_1 = \{(x_1, x_2, x_3) : 2x_1 - x_2 + x_3 \le 4\} \text{ and } H_2 = \{(x_1, x_2, x_3) : x_1 - 2x_2 + x_3 \le 1\}.$$

Since H_1 and H_2 are convex, so $S = H_1 \cap H_2$ is also convex.

Example : Let A be an $m \times n$ matrix and b an m-vector, then show that $\{x \varepsilon R^n : Ax \le b\}$ is a convex set.

Solution : Let $x = (x_1, x_2, ..., x_3)'$, $b = (b_1, b_2, ..., b_3)'$ and $A = (a_{ij})_{m \times n}$. Then the set $S = \{x \varepsilon R^n : Ax \le b\}$ can be represented by m-inequalities :

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$$

Thus, the set S is the intersection of m half spaces,

$$H_i = \{(x_1, x_2, x_n) : a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i, i = 1, 2, \ldots, m\}$$

Since each half space is convex, $S = \bigcap_{i=1}^{m} H_i$ is also convex.

Theorem 4. Given that S and T be two convex sets in R^n, then show that $\alpha S + \beta T$ is also convex α, β.

Proof. Let $S \subset R^n$ and $T \subset R^n$ be two convex sets; and α, $\beta \varepsilon R$.

Let X, Y be two points of $\alpha S + \beta T$.

Then $X = \alpha U_1 + \beta V_1$ and $Y = \alpha U_2 + \beta V_2$ where $U_1, U_2 \varepsilon S$, and $V_1, V_2 \varepsilon T$. ... (1)

For any scalar λ, $0 \le \lambda \le 1$, we have

$$\begin{aligned} \lambda X + (1-\lambda)Y &= \lambda(\alpha U_1 + \beta U_1) + (1-\lambda)(\alpha U_2 + \beta V_2) \\ &= \alpha(\lambda U_1 + (1-\lambda)U_2)] + \beta[\lambda V_1 + (1-\lambda)V_2] \quad \ldots (2) \end{aligned}$$

Since S is a convex set, $U_1, U_2 \varepsilon S \Rightarrow \lambda U_1 + (1-\lambda)U_2 \varepsilon S$, $0 \le \lambda \le 1$, ... (3)

And similarly, $V_1, V_2 \varepsilon T \Rightarrow \lambda V_1 + (1-\lambda)V_2 \varepsilon T$, $0 \le \lambda \le 1$, ... (4)

Now from (1), (3) and (4) $\lambda X + (1-\lambda) Y \varepsilon \alpha S + \beta T$, $0 \le \lambda \le 1$.

Hence $\alpha S + \beta T$ is a convex set.

Corollary. If s and T be two convex sets in R^n, then $S + T$ and $S - T$ are also convex.

Theorem 5. The set of all convex combmations of a finite number of points $x_1, x_2, \ldots, x_m$ is a convex set.

Proof. Let $C = \left\{ x | x = \sum_{i=1}^{m} \lambda_1 x_x, \lambda \ge 0, \sum_{i=1}^{m} \lambda_1 = 1 \right\}$

To show that C is a convex set :

Let x' and x'' be in C, so that

$$x' - \sum_{i=1}^{m} \lambda'_i x_i \left(\text{where } \lambda'_i \geq 0, \sum_{i=1}^{m} \lambda' = 1 \right) \text{ and}$$

$$x'' - \sum_{i=1}^{m} \lambda''_i x_i \left(\text{where } \lambda''_i \geq 0, \sum_{i=1}^{m} \lambda'' = 1 \right)$$

Now consider the vector : $x = \lambda x' + (1-\lambda)x'', 0 \leq \lambda \leq 1$

$$= \lambda \sum_{i=1}^{m} \lambda'_i x_i + (1-\lambda) \sum_{i=1}^{m} \lambda'' x_i$$

$$= \sum_{i=1}^{m} (\lambda\lambda'_i + (1-\lambda)\lambda''_i) x_i = \sum_{i=1}^{m} \mu_i x_i$$

where $\mu_i = \lambda\lambda'_i + (1-\lambda)\lambda''_i$, $i = 1, 2, ..., m$.

Since $0 \leq \lambda \leq 1$, $\lambda_i \leq 0$, we have $\mu_i \geq 0 \ \forall \ i - 1, 2, ..., m$. Also,

$$\lambda \sum_{i=1}^{m} \mu\lambda = \sum_{i=1}^{m} \{\lambda\lambda'_i + (1-\lambda)\lambda''_i\} = \lambda \sum_{i=1}^{m} \lambda'_i + (1-\lambda) \sum_{i=1}^{m} \lambda''_i = 1.$$

Hence x is a convex combination of the points $x_1, x_2, ..., x_m$ in C i.e., $x \varepsilon C$.

Fundamental theorem of linear programming

Consider the LP problem

minimize $z = c'x$

subject to $Ax = b, x \geq 0$.

Let S be the set of feasible solutions to the problem.

Theorem 6. S is a convex set.

Let $x_{(2)}$ be two feasible solutions to the problem, then

$$Ax_{(1)} = b, x_{(1)} \geq 0, Ax_{(2)} = b, x_{(2)} \geq 0.$$

Consider $\quad x = \lambda x_{(1)} + (1-\lambda)x_{(2)}, 0 \leq \lambda \leq 1$.

As $x_{(1)}, x_{(2)} \geq 0$ and $\lambda \geq 0, x \geq 0$.

Also, $Ax \quad = \lambda x_{(1)} + (1-\lambda)Ax_{(2)}$

$= \lambda b + (1-\lambda)b = b.$

Hence x is a feasible solution to the problem, i.e. $x \varepsilon S$.

Thus, S is a convex set.

Theorem 7. Extreme points of S correspond to basic feasible solutions.

Proof. Let $x^0 = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$ be a basic feasible solution to the problem, where x_B is a $m \times 1$ vector satisfying for some $m \times m$ submatrix B of A, $Bx_B = b$. If possible let x^0 be a point of S, i.e. we can find two distinct points $x_{(1)}$ and $x_{(2)}$ in S such that

$$x^0 = \lambda x_{(1)} + (1-\lambda)x_{(2)} \text{ for some } 0 < \lambda < 1.$$

Let $x_{(1)} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $x_{(2)} = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$

where u_1 and u_2 are $m \times 1$ vectors

Then

$$x_B = \lambda u_1 + (1-\lambda)u_2 \quad \text{...(1)}$$

$$0 = \lambda v_1 + (1-\lambda)v_2 \quad \text{... (2)}$$

Since $x_{(1)}$ and $x_{(2)}$ are feasible solutions, $u_1, u_2 \geq 0$, $v_1, v_2 \geq 0$.

Hence, since $0 < \lambda < 1$, it follows from (2) that

$$v_{(1)} = 0,\ v_{(2)} = 0.$$

Therefore, $x_{(1)} = \begin{pmatrix} u_1 \\ 0 \end{pmatrix}$ and $x_{(2)} = \begin{pmatrix} u_2 \\ 0 \end{pmatrix}$

Since $x_{(1)}$ and $x_{(2)}$ satisfy $Ax = b$, we have $Bu_1 = b$, $Bu_2 = b$, so that,

$$u_1, u_2 = B^{-1}b.$$

But $Bx_B = b$, which implies $x_B = B^{-1}b$.

Hence, $u_1 = u_2 = x_B$.

Thus $x_{(1)} = x_{(2)} = x^0$. This contradicts that $x_{(1)} \neq x_{(2)}$.

Hence, x^0 is an extreme point of S.

Theorem 8. If S be a convex polyhedron, then at least one extreme point is an optimal solution.

Proof. Let $x_{(1)}, x_{(2)}, \ldots, x_{(k)}$, be the extreme points of S. Let $x_{(m)}$, $1 \leq m \leq k$ be such that

$$\min_{1 \leq i \leq k} = c' x_{(m)} = z^*, \text{ say.}$$

Let $x_{(0)}$ be a point in S, which is not an extreme point, at with the objective function is minimized, i.e., $\min_{x \in S} c'x_{(0)} = z_0$, say

Since $x_{(0)}$ is not an extreme point and S is a convex polyhedron, $x_{(0)}$ can be expressed as a convex combination of the extreme points $x_{(1)}, x_{(2)}, \ldots, x_{(k)}$, i.e.,

$$x = \lambda_1 x_{(1)} + \lambda_2 x_{(2)} + \ldots + \lambda_k x_{(k)}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$, $\sum_{i=1}^{k} \lambda_1 = 1$.

Therefore,

$$\begin{aligned} z_0 &= c'x_{(0)} \\ &= \lambda_1 c'x_{(1)} + \lambda_2 c'x_{(2)} + \ldots + \lambda_k c'x_{(k)} \\ &\geq \lambda_1 z^* + \lambda_2 z^* + \ldots + \lambda_k z^* \\ &= z^*. \end{aligned}$$

Since z_0 is the minimum value of the objective function is S, we must have

$$z^* = z_0.$$

This implies that there exists an extreme point at which the objective function is minimum.

Theorem 9. If S be a convex polyhedron and the minimum value of the objective function be attained at more than one extreme point of S, then the minimum value of the objective function is attained at every point which is a convex combination of those extreme points.

Proof. Let $x_{(1)}, x_{(2)}, \ldots, x_{(k)}$ be the extreme points of S and without loss of generality, let the objective function assume its minimum value at $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$, $r \leq k$. This means

$$\begin{aligned} c'x_{(1)} &= c'x_{(2)} = \ldots = c'x_{(r)} \\ &= \min_{x \in S} z^*, \text{ say,} \end{aligned}$$

Let $x = \lambda_1 x_{(1)} + \lambda_2 x_{(2)} + \ldots + \lambda x_{(r)}$

where $\lambda_1, \lambda_2, \ldots, \lambda_r \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, be a convex combination of $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$.

Then,

$$\begin{aligned} c'x &= c'\{\lambda_1 x_{(1)} + \lambda_2 x_{(2)} + \ldots + \lambda_n x_{(r)}\} \\ &= \lambda_1 c'x_{(1)} + \lambda_2 c'x_{(2)} + \ldots + \lambda_r c'x_{(r)}. \end{aligned}$$

Hence the theorem.

Theorems 7 and 8 show that if the set of feasible solutions is a convex polyhedron, then we can search for the optimal solution from among the basic feasible solutions. This is the idea behind simplex method.

2.8 Development of the Simplex Method

Consider the LP problem in the standard from, i.e.,

Minimize $z = c'x$

subject to $Ax = b$

$x \geq 0$

Let us write $A - (a_1, a_2, ..., a_n)$, where a_j, $a_{m \times 1}$ vector, is the j-th column of A. Without loss of generality, let

$B = (a_1, a_2, ..., a_m)$ be a non-singular matrix. Since rank $(A) = m$, the columns of B therefore from a basis of the column space of A so that every column of A can be expressed as a linear combination of $a_1, a_2, ..., a_m$. B is called the basis matrix of A and the columns of B are called basic vectors. Let us write

$$a_j = y_{1j}b_1 + y_{2j}b_2 \dots + y_{mj}b_m$$

or, $a_j = By_j$

or, $y_j = B^{-1}a_j$,

where $y_j = (y_{1j}\, y_{2j} \dots y_{mj})', j = 1, 2, ..., n.$

We partition A and x as

$$A = (B \mid A_1), x = \begin{pmatrix} x_B \\ x_R \end{pmatrix}$$

where $A_1 = (a_{m+1}, ..., a_n), x_n = (x_1, ..., x_m)', x_R = (x_{m+1}, ..., x_n)'$

The columns of A_1 are called non-basic vectors.

Then the constraints $Ax = b$ give $Bx_B + A_1x_R = b$.

Putting $x_R = 0$ we get $Bx_B = b$ so that $x_B = B^{-1}b$. This is a b.f.s. to the problem.

since $a_j = y_{1j}b_1 + y_2 \dots + y_{mj}b_m$, if $y_{ij} \neq 0$, we can write

$$b_i = -\sum_{k \neq i} \left(\frac{y_{kj}}{y_{ij}} \right) b_k + \frac{1}{y_{ij}} a_j$$

so that $(b_1, b_2, ..., b_{i-1}, a_j, b_{i+1}, ..., b_m)$ forms a new basis matrix of A.

Accordingly,

$$b = Bx_B = \sum_{k \neq i} \left(x_{Bk} \frac{y_{kj}}{y_{ij}} x_{Bi} \right) b_k + \frac{x_{Bi}}{y_{ij}} a_j.$$

Thus we get a new basic solution is $\hat{x}_B$, having as its components the basic variables

$$\hat{x}_{Bi} - x_{Bi} - \frac{y_{ij}}{y_{rj}} x_{Br}, \, i \, 1 = 1, 2, ..., m, \, i \neq r$$

$$\frac{x_{Br}}{y_{rj}}, \, i = r.$$

We shall now show that $\hat{x}_B$ satisfies the non-negativity restrictions.

Case 1 : Suppose $x_{Br} = 0$. In this case $\hat{x}_{Bi}$ *'s* sill obviously be non-negative.

Case 2 : Suppose $x_{Br} > 0$. In this case we must have $y_{rj} > 0$. To make $\hat{x}_{Br}$ non-negative. Also, for the remaining y_{ij} $(i \neq r)$ we require that for each $i \neq r$, either $y_{ij} = 0$ or

$$\frac{x_{Bi}}{y_{ij}} \geq \frac{x_{Br}}{y_{rj}} \text{ for } y_{ij} > 0,$$

and $$\frac{x_{Bi}}{y_{ij}} \leq \frac{x_{Br}}{y_{rj}} \text{ for } y_{ij} < 0.$$

So, if we select the index r with $y_{rj} \neq 0$ in such a way that

$$\frac{x_{Bi}}{y_{rj}} = \min \left\{ \frac{x_{Bi}}{y_{ij}}, y_{ij} > 0 \right\}$$

then the new set of basic variables $\hat{x}_{Bi}$*'s* will be non-negative, and hence the basic solution $\hat{x}_B$ is feasible. This can be always done.

Remark : If we replace a basic vector b_r from the basis matrix $B = (b_1, b_2, ..., b_m)$ by a non-basic matrix a_j, then the new basis matrix is

$$\hat{B} = (\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_m),$$

where $\hat{b}_i = b_i, i = r$

$\hat{b}_r = a_j,$

and the new b.f.s. is

$$\hat{x}_B = \hat{B}^{-1}b$$

where $\hat{x}_{Bi} = x_{Bi} - \dfrac{y_{ij}}{y_{rj}} x_{Br}, i \neq r$

$$= \frac{x_{Br}}{y_{rj}}, i = r,$$

are the new basic variables.

Theorem 10. Let x_B be a b.f.s. to the LP problem (2.4)-(2.6). Let $\hat{x}_B$ be another b.f.s. obtained by introducing the non-basic vector a_j in the basis for which $z_j - c_j$ is positive. Then $\hat{x}_B$ is an improved b.f.s. the problem in the sense that

$$\hat{c}'_B \hat{x}_B < c'_B x_B$$

Proof. Let $c_B' x_B = z_0$.

Let $\underline{a}_j$ be the non-basic vector admitted in the basis in place of $\underline{b}_r$ and $z_j - c_j > 0$.

If $\hat{x}_B$ be the new b.f.s. then

$$\hat{x}_{Bi} = x_{Bi} - \frac{y_{ij}}{y_{rj}} x_{Br}, i \neq r$$

$$= \frac{x_{Bi}}{y_{rj}}, i = r.$$

And, $\hat{c}_{Bi} = c_{Bi}, i \neq r$

$$= c_j, i = r.$$

Then the value of the objective function corresponding to $\hat{x}_B$ is

$$\hat{z} = \sum_{i=1}^{m} \hat{c}_{Bi} \hat{x}_{Bi}$$

$$= \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{Bi} \left(x_{Bi} - \frac{y_{ij}}{y_{rj}} x_{Br} \right) + c_j . \frac{x_{Br}}{y_{rj}}$$

$$= \sum_{i=1}^{m} c_{Bi}\left(x_{Bi} - \frac{y_{ij}}{y_{rj}} x_{Br}\right) + c_j . \frac{x_{Br}}{y_{rj}}$$

$$= \sum_{i=1}^{m} c_{Bi} x_{Bi} - (z_j - c_j)\frac{x_{Br}}{y_{rj}} < z_0,$$

since $z_j - c_j > 0$.

Hence the new b.f.s. $\hat{x}_B$ gives an improved value of the objective function.

Corollary : If $z_i - c_j = 0$ for at least one j for which $y_{ij} > 0$, $i = 1, 2, ..., m$, then another b.f.s. is obtained which gives an unchanged value of the objective function.

Proof. It follows from theorem 10 that

$$\hat{z} = z_0 - (x_i - c_j)\frac{x_{Br}}{y_{rj}}.$$

$$= z_0, \text{ since } z_1 - c_j = 0, \frac{x_{Br}}{y_{rj}} > 0.$$

Theorem 11. If for a b.f.s. to the LP problem (2.4)-(2.6) there exists at least one j for which $y_{ij} \le 0$ for all $i = 1, 2, ..., m$, but $z_i - c_j > 0$, then there does not exist an optimum solution to the problem, i.e., the problem has an unbounded solution.

Proof. Let $\underline{x}_B$ be a b.f.s. to the problem corresponding to the basis matrix B(i.e. $Bx_B = b$) and having the value of objective function

$$z_0 = c'_B x_B = \sum_{i=1}^{m} c_{Bi} x_{Bi}$$

Now, we can write

$$b = Bx_B + \xi a_j - \xi a_j, \ \xi \text{ being a scalar}$$

$$= \sum_{i=1}^{m} x_{Bi} b_i + \xi a_j - \xi \sum_{i=1}^{m} y_{ij}\, b_i$$

$$= \sum_{i=1}^{m} (x_{Bi} - \xi y_{ij}) b_i + \xi a_j$$

If $\xi > 0$ then $(x_{Bi} - \xi y_{ij}) \ge 0$ since $y_{ij} \le 0$. This shows that there exists a feasible solution whose $(m + 1)$ components may be strictly positive. The corresponding value of the objective function is

$$\hat{z} = \sum_{i=1}^{m} c_{Bi}(x_{Bi} - \xi y_{ij}) + \xi c_j$$

$$= \sum_{i=1}^{m} c_{Bi} x_{Bi} - \xi\left(\sum_{i=1}^{m} c_{Bi} y_{ij} - c_j\right)$$

$$= z_0 - \xi(z_j - c_j)$$

But $\quad z_i - c_j > 0.$

$\therefore \quad \hat{z} < z_0$ for $\xi > 0.$

Hence, $\hat{z} \to -\infty$ as $\xi \to \infty$.

This means that z can be made arbitrarily small. Hence the problem has an unbounded solution.

Remark 1. From the above discussion it follows as long as $z_j - c_j > 0$ and $y_{ij} > 0$ for at least one j, $i = 1, 2, ..., m$, one can get a new b.f.s. which improved the value of the objective function.

Remark 2. If $z_j - c_j < 0$ for all j then the corresponding b.f.s. is optimal.

Proof. Suppose $x_B = B^{-1}b$ is the b.f.s. and $z_i - c_i \leq 0$ corresponding to all $j = 1, 2, .., n$. Consider any arbitrary feasible solution x of $Ax = b$. Since $x_j \geq 0$ for all $j = 1, 2, .., n$.

$$\sum_{j=1}^{n} (z_j - c_j)x_j \leq 0$$

or, $$\sum_{j=1}^{n} z_j x_j \leq \sum_{j=1}^{n} c_j x_j$$

or, $$\sum_{j=1}^{n} (c'_B y_j) x_j \leq \sum_{j=1}^{n} c_j x_j$$

or, $$\sum_{i=1}^{n} c_{Bi}\left(\sum_{j=1}^{n} y_{ij} - x_j\right) \leq \sum_{j=1}^{n} c_j x_j \qquad \text{... (1)}$$

Now, $$\begin{aligned} x_B &= B^{-1}b \\ &= B^{-1}(Ax) \\ &= B^{-1}(a_1, a_2, ..., a_n)x \\ &= (B^{-1}a_1, B^{-1}a_2, ..., B^{-1}a_n)x \\ &= (y_1, y_2, ..., y_n)x \end{aligned}$$

$\therefore \quad x_B = \sum_{i=1}^{n} y_{ji} x_j,\ i = 1, 2, ..., m.$

Hence, from (1) we get

$$\sum_{i=1}^{n} c_{Bi} x_{Bi} \leq \sum_{i=1}^{n} c_j x_j$$

or, $z_0 \leq z^*$,

where z_0 is the value of the objective function corresponding to b.f.s. $\underline{x}_B$ and z^* is the value of the objective function corresponding to an arbitrary feasible solution of $Ax = b$. Thus, z_0 is the minimum value of the objective function.

We are now in a position to write down the computational procedure of the simplex method.

2.9 The Computational Procedure

The optimal solution to the LP problem in the standard from is obtained in the following steps :

Step 1 : Obtain an initial b.f.s. to the problem in the form

$$x_B = B^{-1}b.$$

Step 2 : Compute $z_i - c_j$ for $j = 1, 2, ..., n$ by using the relation

$$z_j - c_j = c_B y_j - c_j,$$

where $y_j = B^{-1}a_j$.

Examine the sign of $z_j - c_j$.

(i) If all $z_j - c_j \leq 0$, the b.f.s. $\underline{x}_B$ is optimal.

(ii) If $z_j - c_j > 0$ for at least one j, go to step 3.

Step 3 : If there is more than one j with $z_j - c_j > 0$, choose the most positive of them, i.e., choose $j = k$, where

$$z_k - c_k = \max_{1 \leq j \leq n} \{z_j - c_j, z_i - c_j > 0\}$$

(i) $y_{ik} \leq 0$ for all $i = 1, 2, ..., m$.

If $a_j = b_i$, then

$$y_j = B^{-1}a_j$$
$$= B^{-1}b_i$$

$= e_i$, the unit vector with i-th element unity and all other elements zero,

$$
\begin{aligned}
z_i - c_j &= c'_B y_j - c_j \\
&= c'_B e_j - c_j \\
&= c_{Bi} - c_j \\
&= 0,
\end{aligned}
$$

since we have $c_{Bi} = c_j$ as $b_i = a_j$.

Then the problem has an unbounded solution.

(ii) If $y_{ik} > 0$ for some $i = 1, 2, ..., m$, then the non-basic vector a_k enters the basis.

Step 4 : Find $i = r$ such that

$$\frac{x_{Br}}{y_{rk}} = \min_{1\le i\le m}\left\{\frac{x_{Br}}{y_{rk}}, y_{ik} > 0\right\}$$

Then b_r leaves the basis.

Step 5 : Compute the new b.f.s. $\hat{x}_B$ with

$$\hat{x}_{Bi} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 01 \end{bmatrix}\begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

$$= \frac{x_{Br}}{y_{rk}}, i = r.$$

Step 6 : Compute the new values of y_{ij}'s from the relation

$$
\begin{aligned}
\hat{y}_{ij} &= y_{ij} - \frac{y_{rj} y_{ik}}{y_{rk}}, i \neq r \\
&= \frac{y_{rj}}{y_{rk}}, i = r
\end{aligned}
$$

where $j = 1, 2, ..., n$.

Step 7 : Go to step 2 and repeat the computational procedure until either optimal solution is obtained or there is an indication of unbounded solution.

Example : Use the simplex method to solve the following LPP :—

$$\text{maximize } z = 7x_1 + 5x_2$$

$$\text{subject to} \quad x_1 + 2x_2 \leq 6$$

$$4x_1 + 3x_2 \leq 12$$

$$x_1, x_2 \geq 0.$$

Solution : We first reduce the given problem to the standard form—

$$\text{maximize } z^* = -7x_1 - 5x_2$$

$$\text{subject to} \quad x_1 + 2x_2 + x_3 = 6$$

$$4x_1 + 3x_2 + x_4 = 12$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

(Have x_3 and x_4 are the slack variables).

The set of equations can be written as

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

or, $\quad Ax = b.$

clearly rank $(A) = 2$ and since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are two linearly independent column vectors A, we can take $B = \begin{bmatrix} 10 \\ 01 \end{bmatrix}$ as a non-singular basis matrix of A. The basic variables are, therefore, x_3 and x_1 and an obvious basic feasible solution is

$$x_B = B^{-1}b$$

$$= \begin{bmatrix} 10 \\ 01 \end{bmatrix} \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

i.e., $\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$

Corresponding to these basic variables, the matrix

$Y = B^{-1}A$, where $Y = (y_1, y_2, ..., y_n)$, and

$$z_j - c_j = c'_B y_j - c_j; j = 1, 2, ..., n$$

are now computed, where c_B is the matrix of costs corresponding to the basic variables in the objective function. Now, for this initial basic feasible solution

$$z = c'_B x_B \quad \text{(since the remaining } x'_j\text{s are zero)}$$

$$= \sum_{i=3,4} c_{Bi} . x_{Bi}$$

$$= 0.$$

To see whether there exists some better basic feasible solution, we compute $z_j - c_j$ for the non-basic variables x_1 and x_2 as follows :

$$z_1 - c_1 = c'_B y_1 - c_1$$

$$= (0 \ \ 0)\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 7$$

$$= 7$$

$$z_2 - c_2 = c'_B y_2 - c_2$$

$$= (0 \ \ 0)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 5$$

$$= 5$$

Thus the initial simplex table is

$$c = (-7 \ -5 \ \ 0 \ \ 0)$$

c_B	B	x_B	y_1	y_2	y_3	y_4	
0	$b_1 = a_3$	$x_3 = 6$	1	2	1	0	$x_{B1}/y_{11} = 6/1 = 6$
0	$b_2 = a_4$	$x_4 = 12$	4*	3	0	1	$x_{B2}/y_{22} = 12/4 = 3$
		$z^* = 0$	7	5	0	0	$z_j - c_j$

Now, since more than one $z_j - c_j$ are positive, we choose the most positive of these, viz., 7, which lies in the column y_1. Since all the components of y_1 are positive, therefore the vector a_1 will enter the basis B.

To select the vector which should leave the basis, we compute $\left\{\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0, i = 1,2\right\}$

and choose the minimum of these ratios, viz., $\frac{12}{4} = 3$. Thus the vector $b_2 = a_4$ leaves the basis. The leading common element is 4, which becomes the leading element for the next iteration. The leading element has been shown in the simplex table in bold type with a star.

First iteration. Introduce a_1 and drop a_4 from B. Convert the leading element to unity by dividing that row by 4 and all other members of the column y_1 to zero by using the relations given is Step 6. Compute again the values $z_j - c_j$.

The next simplex table is obtained as follows :

$$c = (-7 \ -5 \ 0 \ 0)$$

c_B	B	x_B	y_1	y_2	y_3	y_4	
0	$b_1 = a_3$	$x_3 = 3$	1	5/4	1	– 1/4	
–7	$b_2 = a_1$	$x_4 = 3$	1	3/4	0	1/4	$z_j - c_j$
		$z^* = 21$	0	– 1/4	0	– 7/4	

It is apparent from the table that all new $z_j - c_j$ are ≤ 0 and hence an optimum solution has been reached. Thus an optimum basic feasible solution to the given LPP is

$$x_1 = 3, x_2 = 0, \min z^* = -21, \quad \text{or } \max z = 21.$$

2.10 Artificial Variables

Give a LP problem, sometimes it may not be easy to read off a basis matrix B from the coefficient matrix A. In such a situation we add some new variables to the problem called artificial variables to easily get a basis matrix.

Suppose the problem is of the form :

$$\begin{aligned} &\text{minimize } z = c'x \\ &\text{subject to } \quad Ax = b \\ &\qquad\qquad\qquad x \geq 0 \end{aligned}$$

where A is a $m \times n$ matrix of rank $m(\leq n)$..

To the i-th constraint defined in $Ax = b$ let us add a variable $x_{ai} (\geq 0)$ with coefficient 1, for $i = 1, 2, ..., m$. Then the new set of constraints can be written as

$$Ax + 1_m x_a = b$$
$$x \geq 0, x_a \geq 0$$

where I_m is a $m \times m$ identity matrix and $x_a = (x_{a1}, x_{a2}, ..., x_{am})'$.

Or, we can write

$$A^*x^* = b, x \geq 0.$$

where $A^* = (A \; I_m)$

$$x^* = \begin{pmatrix} x \\ x_a \end{pmatrix}$$

Then we obtain $B = I_m$ as a basis matrix of A^*.

Here x_{ai}'s are called artificial variables.

When artificial variables are used in a problem, there are two methods by which we can solve the problem.

Method 1. (Charnes Method of Penalties) : Charnes suggested that a very high cost be attached to each of the artificial variables so that the objection function becomes

$$z^* = z + M(x_{a1} + x_{a2} + ... + x_{am})$$

The reason is that the purpose of introducing artificial variables is to get an initial b.f.s. and so we would like to get rid of these variables once the purpose is served. This method of solving a LP problem is called the Big M Method or the Method of Penalties. After introducing artificial variables and attaching large cost M to each of them, we solve the problem by the Simplex Method.

When we obtain the optimum basic feasible solution, we may be in one of the following situations—

(a) The optimum b.f.s. does not have any artificial variable as a basic variable. In this case every artificial is zero and hence the solution is optimal for the original problem.

(b) The optimum b.f.s. includes an artificial variable at the zero level. In this case we get a degenerate optimal b.f.s.

(c) The optimal b.f.s. includes an artificial variable with a positive value. Hence the given LP problem does not possess an optimum b.f.s. In such a case we say that the given problem has a Pseudo-Optimum b.f.s.

Note : While applying the simplex method whenever a coefficient vector corresponding to some artificial variable leaves the basis, we drop the corresponding y_j vector and all entries relating to that vector from the simplex table.

Example : Use Charnes penalty method to

$$\text{minimize } z = 2x_1 + x_2$$
$$\text{subject to } 3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

Solution : The inequalities representing the constraints for the given problem are reduced to equations by introducing surplus and slack variables $x_3 \geq 0$ and $x_4 \geq 0$, respectively. Further, in order to easily get a basis matrix, two artificial variables $x_5 \geq 0$ and $x_6 > 0$ are also introduced. The problem then becomes

$$\text{minimize } z^* = 2x_1 + x_2 + Mx_5 + Mx_6$$
$$\text{subject to } Ax = b, x \geq 0,$$

where

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 1 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$x = (x_1, x_2, ..., x_6)'$ and $b = (3\ 6\ 3)'$

An obvious basic feasible solution to the problem is thus

$$x_B = (3\ 6\ 3)' = (x_6, x_5\, x_4)'$$

taking $B = I_3$.

Starting table. After computing $Y = (y_1, y_2, ..., y_6)'$ and $z^*_j - c_j, j = 1, 2, ..., 6$, the initial simplex table is

			2	1	0	0	M	M	
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	y_{16}	
M	a_6	3	3*	1	0	0	0	1	
M	a_5	6	4	3	-1	0	1	0	
0	a_4	3	1	2	0	1	0	0	
		$9M$	$7M-2$	$4M-1$	M	0	0	0	$z^*_j - c_j$

Second interation. Introduce a_2 and drop a_5.

			2	1	0	0	
c_B	B	x_B	y_1	y_2	y_3	y_4	
2	a_1	3/5	1	0	1/5	0	
1	a_2	6/5	0	1	– 3/5	0	
0	a_4	0	0	0	1	1	
		– 12/5	0	0	– 1/5	0	$z^*_j - c_j$

Since $z_j - c_j \leq 0$ for all j, and optimal solution to the problem has been attained. Thus the optimum b.f.s. is

$$x_1 = 3/5, x_2 = 6/5$$

and $\min z = 12/5$.

Example 2. Maximize $z = 3x_1 + 2x_2$

subject to $2x_1 + x_2 \leq 2$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0.$$

Solution : After introduction of slack surplus and artificial variables the problem becomes

minimize $z^* = -3x_1 - 2x_2 + Mx_5$

subject to $2x_1 + x_2 + x_3$

$$3x_1 + 4x_2 - x_4 + x_5 = 12$$

$$x_j \geq 0, j = 1, 2, ..., 5.$$

Starting table.

			– 3	– 2	0	0	M	
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	
0	a_3	2	2	1	1	0	0	
M	a_5	12	3	4	0	– 1	1	
		12M	$3M + 3$	$3M + 2$	0	$-M$	0	$z^*_j - c_j$

First iteration.

			-3	-2	0	0	M	
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	
-2	a_2	2	2	1	1	0	0	
M	a_5	4	-5	0	4	-1	1	
		$4M-4$	$-5M-1$	0	$-4M-2$	$-M$	0	$z^*_j-c_j$

Since $z_j - c_j \leq 0$ for all i, we have reached the optimal b.f.s. But this solution has the artificial variable x_5 at a positive level, i.e., $x_5 = 4$. Hence the given LPP does not possess an optimum b.f.s.

2. **Two-phase method :** This is alternative method to solve a LP problem in which artificial variables have been introduced.

The procedure is as follows—

Phase 1. In the first phase we use the simplex method to solve the auxiliary LP problem

$$\text{minimize } z^* = \Sigma x_{ai}$$

subject to the given constraints altered after introduction of artificial variables x_{ai}'s.

The following three cases may arise—

Case 1. min $z^* > 0$. In this case at least one artificial variable is positive. So the original LP problem does not possess any feasible solution. Hence we stop here.

Case 2. min $z^* = 0$ and at least one artificial variable appears in the b.f.s. with value zero.

In such a case we continue with simplex interations with an attempt to drive out all artificial variables from the b.f.s. If we succeed, we get an initial b.f.s. to the original problem and we proceed to phase 2. On the other hand if the artificial variable x_{as} corresponding to the s-th constraint cannot be removed from the basis, the s-th constraint of the original problem is redundant.

Case 3. min $z^* = 0$ and no artificial variable appears in the optimum b.f.s.

In such a case the optimum b.f.s. of the auxiliary problem serves as an initial b.f.s. for the original problem and we proceed to phase 2.

So long as phase 1 ends in min $z^* = 0$, the final simplex table of phase 1 is converted to an initial simplex table of phase 2 by deleting the non-basic artificial vectors and recomputing value of the objective function and $z_i - c'_i$s in accordance to the original problem.

Phase 2. We assign actual costs to the legitimate variables (i.e. decision variables of the original problem) and cost zero to each of the artificial variable appearing in the basis at zero level. Then the Simplex Method is applied until an optimum b.f.s. (if any exists) is obtained. This gives the optimal solution to the original LP problem.

Example : Use two-phase simplex method to

$$\text{minimize } z = -3x_1 - 2x_2$$

$$\text{subject to } \quad 2x_1 + x_2 \le 2$$

$$3x_1 + 4x_2 \ge 12$$

$$x_1, x_2 \le 0.$$

Solution : After introducing slack and surplus variables x_3 and x_4 respectively and an artificial variable x_5 to the second constraint we get the constraints as

$$A^*x^* = b$$

$$A^*x^* \ge 0,$$

where $$A^* = \begin{pmatrix} 2 & 1 & 1 & 0 & 1 \\ 3 & 4 & 0 & -1 & 1 \end{pmatrix}$$

$$x = (x_1, x_2, x_3, x_4, x_5)', b = (2, 12)'$$

So an initial b.f.s. is $x_B = \begin{pmatrix} x_3 \\ x_5 \end{pmatrix}$

with basis matrix $I_2 = (a_3 a_5)$.

Phase 1. The auxiliary problem is

$$\text{minimize } z^* = x_5$$

$$\text{subject to } \quad A^*x^* = b$$

$$x^* \ge 0.$$

Starting table.

			0	0	0	0	1	
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	
0	a_3	2	2	1*	1	0	0	
1	a_5	12	3	4	0	– 1	1	
		12	3	4	0	– 1	0	$z^*_j - c_j$

First iteration. Introduce a_2 and remove a_3.

			0	0	0	0	1	
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	
0	a_2	2	2	1	1	0	0	
1	a_5	4	– 5	0	– 4	– 1	1	
		4	– 5		– 4			$z^*_j - c_j$

Since $z_j - c_j \leq 0$ for all j,

$x_B \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is an optimal solution to the auxiliary LP problem. But min $z^* = 4 > 0$. Hence original LP problem does not possess a feasible solution.

Example. Solve the problem

$$\text{minimize } z = -x_1 + x_2$$
$$\text{subject to } \quad 2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

Solution : Adding surplus variables x_3 and x_4 and the artificial variables x_{a1} and x_{a2} to the two constraints be get the constraint equations as

$$2x_1 + x_2 - x_3 + x_{a1} = 4$$
$$x_1 + 7x_2\, x_4 \quad + x_{a2} = 7$$

subject to $x_1, x_2, x_3, x_4, x_{a1}, x_{a2} \geq 0.$

Phase 1. The auxiliary problem is

$$\text{minimize } z^* = x_{a1} + x_{a2}$$
$$\text{subject to } A^*x^* = b$$
$$x^* \geq 0,$$

where $A^* = \begin{pmatrix} 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 7 & 0 & -1 & 0 & 1 \end{pmatrix}$

$x = (x_1, x_2, x_3, x_4, x_{a1}, x_{a2})'$, $b = (4, 7)'$

Starting table.

			0	0	0	0	1	1	
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
1	a_5	$x_{a1} = 4$	2	1	– 1	0	1	0	
1	a_6	$x_{a2} = 7$	1	7	0	– 1	0	1	
		$z^* = 13$	3	8	–	– 1	0	0	$z^*_j - c_j$

First iteration. Introduce a_2 and remove a_6.

			0	0	0	0	1	1	
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
1	a_5	$x_{a1} = 3$	– 13/7	0	– 1	1/7	1	1/7	
0	a_2	$x_2 = 1$	– 1/7	1	0	– 1/7	0	– 1/7	
		$z^* = 3$	– 13/7	0	– 3	1/7	0	– 6/7	$z^*_j - c_j$

We drop column a_6.

Second iteration. Introduce a_4 and remove a_5.

			0	0	0	0	1	
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	
0	a_4	$x_4 = 21$	13	0	7	1	– 7	
0	a_2	$x_2 = 4$	2	1	– 1	0	– 1	
		$z^* = 0$	0	0	0	0	– 1	$z_j - c_j$

So the optimal solution has $x_{a1} = x_{a2} = 0$. We, therefore, drop column y_5 and go to phase 2.

Phase II. The starting table for the simplex method is as follows :

Starting table.

			1	1	0	0	
c_B	B	x_B	y_1	y_2	y_3	y_4	
0	a_4	$x_4 = 21$	13	0	7	1	
1	a_2	$x_2 = 4$	2	1	-1	0	
		$z^* = 4$	1	0	-1	0	$z_j - c_j$

We apply simplex method and finally obtain the optimal solution as

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 21/13 \\ 10/13 \end{pmatrix} \text{ and min } z = 31/13.$$

2.11 Duality in LP Problems

For every LP problem there exists another LP problem called its dual. The original problem is known as the primal problem. Consider the LP problem

$$\left.\begin{aligned} &\text{minimize } z = c'x \\ &\text{subject to } \quad Ax \geq b \\ &\qquad\qquad\qquad x \geq 0 \end{aligned}\right\} \qquad \text{... (1)}$$

where A is a $m \times n$ matrix, x and c are $n \times 1$ vectors, b is $m \times 1$ vector. The dual of the above problem is given by

$$\left.\begin{aligned} &\text{minimize } z^* = b'w \\ &\text{subject to } \quad A'w \leq c \\ &\qquad\qquad\qquad w \geq 0 \end{aligned}\right\} \qquad \text{... (2)}$$

where w is a $m \times 1$ vector, which is the decision vector for the dual problem.

2.12 Duality Theorems

Theorem 1. The dual of the dual problem is the primal problem.

Proof. The dual problem (2) can be written as

$$\text{minimize } z^{**} = -b'w$$
$$\text{subject to } \quad -A'w \geq -c$$
$$w \geq 0$$

So its dual will be

$$\text{maximize } \hat{z} = -c'x$$
$$\text{subject to } \quad -Ax \leq -b$$
$$x \geq 0$$

which may be written as

$$\text{minimize } z = c'x$$
$$\text{subject to } \quad Ax \geq b$$
$$x \geq 0.$$

This is the primal problem (1).

Theorem 2. The value of the objective function $z = c'x$ for any feasible solution of the primal is not less than the value of the objective function $z^* = b'w$ for any feasible solution of the dual.

Proof. Let $z = x_0$ be a feasible solution to the primal problem (1) and w_0 be a feasible solution to the dual problem (2).

Then,

$$Ax_0 \geq b, \quad x_0 \geq 0$$
$$A'w_0 \leq c, \quad w_0 \geq 0.$$

Therefore,

$$c' \geq w'_0 A$$
$$\text{or,} \quad c'x_0 \geq w'_0 A x_0$$
$$\text{or,} \quad c'x_0 \geq w'_0 b = b'w_0.$$

Hence the theorem.

Corollary. It immediately follows that

$$\min c'x \geq \max b'w$$
$$\text{or,} \quad \min z \geq \max z^*.$$

Theorem 3. The minimum value of z in (1), if it exists is equal to the maximum value of z^* in (2).

Proof. Let x_0 and w_0 be the optimum feasible solutions to (1) and (2) respectively.

After introducing surplus variables in (1) we get

$$(A - I_m)x^* = b,\ x^* \geq 0$$

or, $$A^*x^* = b,\ x^* \geq 0.$$

where $x^* = (x_1, x_2, ..., x_{n+m})'$, and x_{n+m} are the surplus variables.

Let $x^*_0 = (x_{10}, x_{20}, ..., x_{n+m,0})'$ be an optimal b.f.s. to the primal problem (1) and $z_0 = \min z$.

The equality constraints are

$$\sum_{i=1}^{n} a_{ij}x_j - x_{n+1} = b_i,\ i = 1, 2, ..., m,$$

so that

$$\sum_{i=1}^{n} a_{ij}x_j = b_i + x_{n+i}.$$

So the objective function can be written as

$$\begin{aligned} z &= c'x \\ &= \sum_{j=1}^{n} c_j x_j \\ &= \sum_{j=1}^{n} \left\{ c_j + \sum_{i=1}^{m} \pi_i a_{ij} \right\} x_j - \sum_{i=1}^{m} \pi_i (b_i + x_{n+i}) \end{aligned}$$

for arbitrary $\pi_1, \pi_2, ..., \pi_m$.

For any b.f.s. let π'_is be so chosen that

$$c_j + \sum_{i=1}^{m} \pi_i a_{ij} = 0 \text{ if } x_j \text{ is a basic variables } j = 1, 2, ..., n$$

and $\pi_i = 0$ if x_{n+i} is a basic variable $i = 1, 2, ..., m$.

Then corresponding to that b.f.s. the value of the objective function is

$$z = -\sum_{i=1}^{m} \pi_i b_i.$$

If we write $A^* = (a^*_1, a^*_2, ..., a^*_{n+m})$ then above choice of π'_i simply that

$$c_j + \pi' a^*_j = 0 \quad \text{if } a_j \text{ is a column of the basis matrix } B,$$

where $c_j = 0$ for $j = n + 1, n + 2, ..., n + m$.

Hence, writing $\pi = (\pi_1, \pi_2, ..., \pi_m)'$,

$$c'_B + \pi' B = 0.$$

which gives $\qquad \pi' = - c'_B B^{-1}$,

Therefore, for any non-basic vector a^*_j,

$$\pi' a^*_j + c_j = - c'_B B^{-1} a^*_j + c_j = - (z_j - c_j)$$

So, for the optimal b.f.s.

$$\min z = - \sum_{i=1}^{m} \pi_i b_i$$

where $\qquad \pi_1, \pi_2, ..., \pi_m$ are such that

$$\pi' a^*_j + c_j \geq 0 \text{ for all } j = 1, 2, ..., m. \qquad ... (3)$$

Since a^*_{n+i} is a unit vector with i-th element -1 and other elements zero, and $c_{m+i} = 0$ for $i = 1, 2, ..., m$, it clearly follows from (1) that

$$- \pi_i \geq 0, i = 1, 2, ..., m \qquad ... (4).$$

The inequalities (3) and (4) mean the $w = (-\pi_i, -\pi_2, ..., -\pi_m)'$ is feasible solution of the dual problem. Further,

$$b'w = - \sum_{i=1}^{m} \pi_i b_i = \min z.$$

Thus we have found a feasible solution to the dual problem for which

$$z^* = \min z.$$

By corollary of theorem 3, this is possible only when

$$b'w = \max z^*$$

Hence, $\min z = \max z^*$.

Theorem 4. If the primal problem has an unbounded solution, the dual problem has no feasible solution, and vice versa.

Proof. Suppose the primal problem (1) has an unbounded solution. Then $\min z = -\infty$. If possible, let the dual problem (2) have a feasible solution w. Then, by theorem 2,

$$b'w \leq -\infty,$$

which is a contradication.

Hence, the dual problem has no feasible solution.

The other part of the theorem can be similarly proved.

From the above discussion we conclude that if we can solve the primal (dual) problem, the optimum value of the objective function of the primal (dual) immediately follows.

In order to get the optimal solution to the dual from the optimal solution to the primal we follow the following rules :

Rule 1. If the primal (dual) variable corresponds to a slack starting variable in the dual (primal) problem, its optimum value is directly read off from the optimum dual (primal) simplex table as the $(z_j - c_j)$ value corresponding to the slack variable.

Rule 2. If the primal (dual) variable corresponds to an artificial starting variable in the dual (primal) problem, its optimum value is directly read off from the optimum dual (primal) simplex table as the $(z_j - c_j)$ value corresponding to the artificial variable, after deleting the large cost M.

Example. Use duality to solve the following LP problem—

$$\text{maximize } z = 3x_1 - 2x_2$$
$$\text{subject to } x_1 + x_2 \leq 5$$
$$x_1 \leq 4$$
$$1 \leq x_2 \leq 6$$
$$x_1, x_2 \geq 0.$$

Solution : The last two constraints of the problem can be written as

$$x_1 + 0x_2 \leq 4,\ 0x_1 + x_2 \leq 6,\ 0x_1 - x_2 \leq -1.$$

The dual of the given problem is, therefore,

$$\text{minimize } z^* = 5w_1 + 4w_2 + 6w_3 - w_4$$
$$\text{subject to } w_1 + w_2 \geq 3$$
$$w_1 + w_3 - w_4 \geq -2$$
$$w_1, w_2, w_3, w_4 \geq 0.$$

Let surplus variables $w_5 \geq 0$ and slack variable $w_6 \geq 0$ be appropriately introduced.

Then an initial b.f.s. is

$$w_B = (w_2, w_4)' = (3, 2)'.$$

Starting table.

c_B	B		5	4	6	-1	0	0	
		w_B	y_1	y_2	y_3	y_4	y_5	y_6	
4	$b_1 = a_2$	$w_2 = 3$	1	1	0	0	1	0	
-1	$b_2 = a_4$	$w_4 = 2$	1	0	1	1	0	1	
	$z^* = 10$		0	0	-5	0	-4	-1	$z_j - c_j$

Since $z_j - c_j \le 0$ for all j, and optimal b.f.s. for the dual problem is obtained. Thus the optimal solution to the dual problem is $w_2 = 3$, $w_4 = 2$, $w_1 = w_3 = 0$, and min $z^* = 10$.

As the primal variables x_1 and x_2 correspond to the slack and surplus variables w_5 and w_6 of the dual problem, the optimal solution to the primal problem will be $x_1 = 4$, $x_2 = 1$, and max $z = 10$.

2.13 Sensitivity Analysis

In many practical situations we may want to find not only the optimal solution to a given problem, but also the effect of changes in the system on this solution. Such investigations are referred to as sensitivity analysis.

For a given LP problem the changes in the system may be classified as follows :

(a) Change in the requirement vector b;

(b) Change in the cost vector c;

(c) Change in the coefficient matrix A;

(d) Introduction of a new variable;

(e) Introduction of a new constraint.

Consider the LP problem

$$\text{minimize } z = c'x$$
$$\text{subject to } Ax = b$$

Let $x_B = B^{-1}b$ be the optimal b.f.s. and z_0 be the corresponding value of the objective function.

(i) Change in b. Let b change to $b^* = b + \Delta b = (b_1 + \Delta b_1, b_2 + \Delta b_2, ..., b_m, ..., b_m + \Delta b_m)'$

Corresponding to the basis B, the basic solution to the new problem, therefore, is

$$\begin{aligned} x^*_B &= B^{-1}b^* \\ &= B^{-1}(b + \Delta b) \\ &= B^{-1}b + B^{-1} + \Delta b \\ &= x_B + B^{-1}\Delta b \\ &= x_B + \sum_{k=1}^{m} (\Delta b_k)\beta_k, \end{aligned}$$

where β_k is the k-th column of B^{-1}.

In order that x^*_B be feasible we must have

$$x^*_{Bi} \geq 0, \quad i = 1, 2, ..., m$$

i.e., $x_{Bi} + \sum_{k=1}^{m} \beta_{ik}\Delta b_k \geq 0$, where $\beta_{ik} = (i, k)$th element of B^{-1}

i.e., $\sum_{k=1}^{m} \beta_{jk}\Delta b_k \geq -x_{Bi}, \quad i = 1, 2, ..., m.$

In particular, if only one component of b changes, say b_k changes to $b_k + \Delta b_k$, then

$$x^*_{Bi} = x_{Bi} \text{ for } i \neq k.$$

$$x^*_{Bi} = x_{Bk} + \beta_{ik}\Delta b_k.$$

So, for feasibility of the new solution we must have

or, $\begin{cases} \Delta b_k \geq -\dfrac{x_{Bk}}{\beta_{ik}} \text{ if } \beta_{ik} > 0. \\ \Delta b_k \leq -\dfrac{x_{Bk}}{\beta_{ik}} \text{ if } \beta_{ik} < 0. \end{cases}$

$$z_j - c_j \leq 0 \text{ for all } j.$$

Since $(z_j - c_j)$-values are unaffected by change in b, the $(z_j - c_j)$ values corresponding to x_B will be same as those corresponding to x^*_B, and as x_B is optimal for the old problem, $z_j - c_j \leq 0$ for all $j = 1, 2, ..., n$. Hence, if x^*_B is feasible, it will be optimal b.f.s. for the new problem.

(ii) **Change in c.** Let c change to $c^* = c + \Delta c$.

Since the constraints are independent of c, x_B will be a b.f.s. for the new problem. The new $(z_j - c_j)$-values will be

$$\begin{aligned} z^*_j - c^*_j &= c^{*\prime}_B B^{-1} a_j - c^*_j \\ &= (c_B + \Delta c_B)' B^{-1} a_j - (c_j + \Delta c_j) \\ &= (z_j - c_j) + (\Delta c_B B^{-1} a_j - \Delta c_j)' \end{aligned}$$

So, x_B will be optimal for the new problem if

$$z^*_j - c^*_j \le 0 \text{ for all } j$$

i.e., $\Delta c'_B B^{-1} a_j - \Delta c_j \le -(z_j - c_j)$ for all j.

(iii) Change in A. Suppose $A = a_1, a_2, ..., a_n)$, and a_k changes to a^*_k.

(a) If a_k a non-basic vector in the optimal solution x_B, i.e., B does not contain a_k, then x_B will be a b.f.s. to the new problem. Now, for $j = 1, 2, ..., n, j \ne k$,

$$(z_j - c_j)_{\text{new}} = (z_j - c_j)_{\text{old}} \le 0$$

and $(z_k - c_k)_{\text{new}} = c'_B B^{-1} a^*_k - c_k$.

So, $(z_k - c_k)_{\text{new}} \le 0$

if $c_k \ge c'_B B^{-1} a^*_k$.

This is the condition for x_B to be optimal for the new problem.

(b) If a_k be a basic vector, things become complicated since the feasibility of the current optimal solution x_B may be destroyed.

Let $B = (b_1, b_2, ..., b_k, ..., b_m)$

and $b_k = a_k$.

Let $B^* = (b_1, b_2, ..., b^*_k, ..., b_m)$

where $b^*_k = a^*_k$.

If B^* is a singular matrix B^{*-1} will not be defined. On the other hand, if B^* is a basis matrix, $x^*_k = B^{*-1} b$ will define a basic solution of the new problem.

If $x^{*-1} b$ will define a basic solution of the new problem.

If $x^*_{Bi} \ge 0$ for all $i = 1, 2, ..., m$, the solution will be also feasible. In that case we check $(z_j - c_j)$-values for the new problem corresponding to b.f.s. x^*_B and if $z_j - c_j \le 0$ for all $j = 1, 2, ..., n$, the solution will be optimal.

(iv) **Addition of a new variable.** Let a new variable x_{n+1} be added to the problem with associated cost c_{n+1} and coefficient vector a_{n+1}. So the new problem is

minimize $z^* = z + c_{n+1}, x_{n+1}$

subject to $A^*x^* = b,$

$x^* \geq 0$

where $A^* = (A\varepsilon_{n+1})$ is a $m \times (n+1)$ matrix

$$x^* = \begin{pmatrix} x \\ x_{n+1} \end{pmatrix}$$

Since B will also be a basis matrix of A^*, x_B will be a b.f.s. to the new problem. To check whether this solution remains optimal we have to check whether

$$z_{n+1} - c_{n+1} = c'_B B^{-1} a_{n+1} - c_{n+1} \leq 0.$$

(v) **Addition of a new constraint.** Suppose a new constraint

$$a_{m+1,1}x_1 + a_{m+1,2}x_2 + \ldots + a_{m+1,n}x_n \leq b_{m+1}$$

is added to the LP problem.

If x_B satisfies the new constraint, x_B remains optimal for the new problem as the extra constraint does not enlarge the feasible region of the LP problem. If x_B does not satisfy the new constraint, we have to search for a new optimal solution.

Let x_s be a slack variable introduced in the new constraint so that the set of constraints becomes

$$Ax = b$$

$$\sum_{i=1}^{n} a_{m+1,j}x_j + x_s = b_{m+1}.$$

Then it can be easily shown that

$$x^*_B = \begin{pmatrix} x_B \\ x_s \end{pmatrix} \text{ will be a b.f.s. to the new problem.}$$

Advantage of linear programming technique

The advantages of linear programming technique may be outlined as follows :

1. Linear programming helps us in making the optimum utilization of production resources. It also indicates how a decision maker can employ

his productive factors most effectively by choosing and allocating these resources.

2. The quality of decisions may also be improved by linear programming technique. The user of this technique becomes more objective and less subjective.

3. Linear programming technique provides practically applicable solutions since there might be other constraints operating outside the problem which must be taken into consideration. Like, just because so many units must be produced does not mean that all those can be sold. So necessary modification of its mathematical solution is required for the sake of convenience to the decision maker.

4. In production processes highlighting of bottlenecks is the most significant advantage of this technique. For example, when bottlenecks occur some machines cannot meet the demand while others remain idle for some time.

2.14 Questions

1. What is a linear programming problem? Define the following in relation to a linear programming problem : (a) feasible solution, (b) optimal solution, (c) basic feasible solution, (d) unbounded solution, (e) degenerate solution, (f) slack variable, (g) surplus variable, (e) artificial variable.

2. A small manufacturer employs 5 skilled men and 10 semi-skilled men for making a product in two qualities : a deluxe model and an ordinary model. The production of a deluxe model requires 2-hour work by a skilled man and 2-hour work by a semi-skilled man. The ordinary model requires 1-hour work by a skilled man and 3-hour work by a semi-skilled man. According to worker union's rules, no man can work more than 8 hours per day. The profit of the deluxe model is Rs. 1000 per unit and that of the ordinary model is Rs. 800 per unit. Formulate a linear programming model for this manufacturing situation to determine the production volume of each model such that the total profit is maximized.

3. The manager of an oil refinery has to decide on the optimal mix of two possible blending processes. The inputs and the outputs per production run of the blending

processes. The inputs and the outputs per production run of the blending process are as follows :

	Input		Output	
Process	Crude A	Crude B	Gasoline G_1	Gasoline G_2
1	5	3	5	8
2	4	5	4	4

The maximum a mounts of a vailability of crude A and B are 200 units and 150 units, respectively. Market requirements show that at least 100 units of gasoline G_1 and 80 units of gasoline G_2 must b produced. The profits per production run from process 1 and process 2 are Rs. 3,00,000 and Rs. 4,00,000, respectively. Formulate this problem as a *LP* model to determine the number of each process such that the total profit is maximized.

4. Solve the following *LP* problems graphically :

(a) minimize $z = 20x_1 + 10x_2$

subject to $x_1 + 2x_2 \leq 40$

$3x_1 + x_2 \geq 30$

$4x_1 + 3x_2 \geq 60$

x_1 and $x_2 \geq 0$.

(b) maximize $z = 60x_1 + 90x_2$

subject to $x_1 + 2x_2 \leq 40$

$2x_1 + 3x_2 \leq 90$

$x_1 - x_2 \geq 10$

x_1 and $x_2 \geq 0$.

5. Solve the following *LP* problem using the simplex method :

maximize $z = 3x_1 + 2x_2 + 5x_3$

subject to $x_1 + x_2 + x_3 \leq 9$

$2x_1 + 3x_2 + 5x_3 \leq 30$

$2x_1 - x_2 - x_3 \leq 8$

$x_1, x_2, x_3 \geq 0$.

6. Solve the following *LP* problem :

maximize $z = 6x_1 + 4x_2$

subject to $2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0.$

7. Solve the following *LP* problem using the two-phase method :

minimize $z = 10x_1 + 6x_2 + 2x_3$

subject to $-x_1 + x_2 + x_3 \geq 1$

$3x_1 + x_2 - x_3 \geq 2$

$x_1, x_2, x_3 \geq 0.$

8. Use duality to solve the following problem :

maximize $z = 2x_1 + x_2$

subject to $x_1 + 2x_2 \leq 10$

$x_1 + x_2 \leq 6$

$x_1 - x_2 \leq 2$

$x_1 - 2x_2 \leq 1$

$x_1, x_2 \geq 0.$

8. Consider the following *LP* problem :

maximize $z = 3x_1 + 2x_2 - 5x_3$

subject to $x_1 + x_2 \leq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 0$

$x_1, x_2, x_3 \geq 0.$

Solve the problem. If the requirement vector is changed to (2, 10, 5), will the optimal solution remain optimal?

9. Solve the following *LP* fproblem :

maximize $z = x_1 + 5x_2 + 3x_3$

subject to $x_1 + 2x_2 + x_3 = 3$

$$2x_1 - x_2 = 4$$

$$x_1, x_2, x_3 \geq 0.$$

If the cost vector is changed to (2, 5, 2), what will be the new optimal solution?

10. Solve the following *LP* problem :

maximize $z = 20x_1 + 80x_2$

subject to $4x_1 + 6x_2 \leq 90$

$$8x_1 + 6x_2 \leq 100$$

$$x_1, x_2 \geq 0.$$

If the following new constraint is added to the model, find the optimal solution to the new problem—

$$5x_1 + 4x_2 \leq 80.$$

Unit 3 ❑ Transportation Problem

Structure

3.1 Introduction

Allocation problems involve the allocation of resources to jobs that need to be done. Such problems arise when the available resources are not sufficient to allow each job to be carried out in the most efficient manner. Therefore, the objective is to allot the resources to the jobs in such a way so as to either minimize the total cost or maximize the total return. When both jobs and resources are expressed in the same units or the same scale, we have what is generally called a **transportation problem**.

A typical transportation problem is one which involves shipment of goods from each of a number of centres called 'origins' to more than one place called 'destinations' and the cost of shipping from each origin to each destination is different and known. The problem is to ship the goods in such a manner that the total cost of transportation is a minimum.

We can formally define a transportation problem as follows :

The transportation problem is to transport various amounts of a single homogeneous commodity at a number of origins to different destinations in such a way that the total transportation cost is a minimum.

A transportation problem can be stated in terms of a $m \times n$ cost matrix, where m denotes the number of origins and n denotes the number of destinations.

3.2 Mathematical Formulation of a Transportation Problem

Let there be m origins and n destinations. Let the i-th origin possess a_i units of a product and b_j be the number of units required by the j-th destination, $i = 1, 2, ..., m, j = 1, 2, ..., n$. If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, i.e., the total availability is equal to the total requirement, then the problem is said to be a balanced transportation problem, else it is called an unbalanced transportation problem.

We shall first consider a balanced transportation problem.

Let c_{ij} be the cost of shipping one unit from origin i to destination j and x_{ij} be the amount shipped from origin i to destination j. Then the problem is to

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to } \quad \sum_{j=1}^{n} x_{ij} = a_i, \; i = 1, 2, ..., m \qquad ... (1)$$

$$\sum_{j=1}^{n} x_{ij} = b_j, \; j = 1, 2, ..., n \qquad ... (2)$$

$$x_{ij} \geq 0 \text{ for } i = 1, 2, ..., m, j = 1, 2, ..., n.$$

Here (1) gives the availability constraints and (2) the requirement constraints, and the objective function z is the total transportation cost.

Clearly, the problem is a LP problem. So to find the optimal solution to the problem we trace our path through the basic feasible solutions.

Theorem 1. (Existence of feasible solution) : A balanced transportation problem always has a feasible solution.

Proof : Let $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j = k$ (say).

Let us define

$x_{ij} = \lambda_i b_j$, for all i and j, where λ_i's are real numbers.

From constraints (1)

$$a_i = \sum_{i=1}^{n} x_{ij} = \lambda_i \sum_{j=1}^{n} b_j = \lambda_i k$$

So, $\lambda_i = \frac{a_i}{k}$, for $i = 1, 2, ..., m$.

Then, for every $j = 1, 2, ..., n$,

$$\sum_{j=1}^{n} x_{ij} = b_j \sum_{j=1}^{m} \lambda_i = \frac{b_j}{k} \sum_{i=1}^{n} a_i = b_j,$$

which means that the x_{ij}'s satisfy constraints (2).

Further,

$$x_{ij} = \lambda_i b_j = \frac{a_i b_j}{k} \geq 0, \text{ since } a_i \geq 0,\ b_j \geq 0 \text{ for all } i \text{ and } j.$$

Hence, the solution is feasible.

Theorem 2. (Existence of an optimal solution) : There always exists an optimal solution to a balanced transportation problem.

Proof. Since $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, there exists a feasible solution to the transportation problem. Further, from constraints (1) and (2) and the non-negativity restrictions it follows that $0 \leq x_{ij} \leq \min (a_i, b_j)$.

Hence, x_{ij}'s are bounded. Thus the region of feasible solutions is closed, bounded and nonempty, and hence an optimal solution exists.

Theorem 3 : Of the $(m+n)$ equations to be satisfied by a feasible solution to the transportation problem, $(m+n-1)$ of them are independent.

Proof : Constraints (1) give

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m$$

and the first $(n-1)$ constraints in (2) are $\sum_{i=1}^{m} x_{ij} = b_j, j = 1, 2, \ldots, n-1$.

$$\text{Therefore,} \quad \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j$$

$$\text{Also,} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$$

$$\text{Again,} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m}\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} + \sum_{i=1}^{m} x_{in}$$

$$\text{or,} \quad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j + \sum_{i=1}^{m} x_{in}$$

$$\text{or,} \quad \sum_{i=1}^{m} x_{in} = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j = b_n \left(\text{since} \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \right)$$

which is the last requirement constraint. This indicates that if the m availability constraints and the first $(n-1)$ requirement constraints be satisfied then $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ ensures that the n-th requirement constraint is satisfied. Hence, out of the $(m+n)$ constraints, one is redundant and the remaining $(m+n-1)$ constraints are independent.

Remark 1 : From theorem 2 it is clear that a basis of the coefficient matrix (as defined in a standard LP problem) will contain $(m+n-1)$ independent columns of A. Hence out of the mn decision variables, a b.f.s. will contain $(m+n-1)$ variables as basic variables. So, a b.f.s. can have at most $(m+n-1)$ positive variables.

Remark 2 : Based on the theorems of linear programming, one of the basic feasible solutions will be an optimal solution to the transportation problem.

3.3 Solving a Transportation Problem

Finding an optimal solution to a transportation problem consists of the following two steps—

(i) finding an initial basic feasible solution

(ii) making successive improvements to initial basic feasible solution until no further decrease in the total transportation cost is possible and thereby obtaining the optimal b.f.s.

3.4 Methods for Initial B.F.S.

Some simple methods are described here to obtain the initial basic feasible solution to the transportation problem. These methods can be easily explained by considering the following numerical example. However, the relative efficiency of these methods is still unanswerable.

Example 1. Find the initial basic feasible solution of the following transportation problem.

Table 3.1

Warehouse Factory 1	W_1	W_2	W_3	W_4	Factory capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse requirement	5	8	7	14	34

Solution :

First method : North-West Corner Rule

In this method, first construct an empty 3×4 matrix complete with row and column requirements (Table 3.2).

Table 3.2

	W_1	W_2	W_3	W_4	Available
F_1					7
F_2					9
F_3					18
	5	8	7	14	

Insert a set of allocation in the cells in such a way that the total in each row and each column is the same as shown against the respective rows and columns. Start with cell (1, 1) at the north-west corner (upper left-hand corner) and allocate as much as possible there. In other words $x_{11} = 5$, the maximum which can be allocated to this cell as the total requirement of this column is 5. This allocation ($x_{11} = 5$) leaves the surplus amount of 2 units for row 1 (Factory F_1). So allocate $x_{12} = 2$ to cell (1, 2). Now, allocations for first row and first column are complete, but there is a deficiency of 6 units in column 2. Therefore, allocate $x_{22} = 6$ in the cell (2, 2). Column 1 and column 2 requirements are satisfied, leaving a surplus amount of 3 units for row 2. So allocate $x_{23} = 3$ in the cell (2,3), and column 3 still requires 4 units. Continuing in this way, from left to right and top to bottom, eventually complete all requirements by an allocation $x_{34} = 14$ in the south-east corner. Table 3.3 shows the resulting feasible solution.

Table 3.3

5(19)	2(30)			7
	6(30)	3(40)		9
		4(70)	14(20)	18
5	8	7	14	

On multiplying each individual allocation by its corresponding unit cost in parenthesis and adding, the total cost becomes = 5(19) + 2(30) + 6(30) + 3(40) + 4(70) + 14(20) = Rs. 1015.

Second method : The Row Minima Method

Step 1. The transportation table of the given problem has 12 cells. Following the row minima method, since min (19, 30, 50, 10) = 10, the first allocation is made in the cell (1, 4), the amount of the allocation is given by x_{14} = min (7, 14) = 7. This exhausts the availability from factory F_1 and thus we cross-out the first row from the transportation table to get Table 3.4.

Table 3.4

	W_1	W_2	W_3	W_4	
F_1				7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	5	8	7	7	

Step 2. In the resulting transportation table, since min (70, 30, 40, 60) = 30, the second allocation is made in the cell (2, 2), the amount of allocation being x_{22} = min(9, 8) = 8. This satisfies the requirement of warehouse W_2 and thus we cross-out the second column from the transportation table obtaining new table 3.5.

Table 3.5

	W_1	W_2	W_3	W_4	
F_1				7(10)	X
F_2	(70)	8(30)	(40)	(60)	1
F_3	(40)				18
	5	X	7	7	

Step 3. In table 3.5 since min (70, 40, 60) = 40, the third allocation is made in the cell (3, 3), the amount being x_{33} = min [1, 7] = 1. This exhausts the availability from factory F_2. And thus we cross-out the second row from the table getting table 3.6.

Table 3.6

	W_1	W_2	W_3	W_4	
F_1				7(10)	X
F_2		8.(30)	1(40)		X
F_3	(40)		(70)	(20)	18
	5		6	7	

Step 4. The next alocation is made in the cell (3,4), since min (40, 70, 20) = 20, the amount of allocation being x_{34} = min(7, 18) = 7. This exhausts the requirement of warehouse W_4 and thus we cross-out the fourth column to get Table 3.7.

Table 3.7

	W_1	W_2	W_3	W_4	
F_1				7(10)	X
F_2		8(30)	1(40)		X
F_3	(40)		(70)	7(20)	11
	5	X	6	X	

Step 5. The next allocation is made in the cell (3, 1), since min (40, 70) = 40, the amount of allocation being x_{31} = min (5, 11) = 5. This satisfies the requirement of warehouse W_1 and so we cross-out the first column W_1 to get new table 3.8.

Table 3.8

	W_1	W_2	W_3	W_4	
F_1				7(10)	X
F_2		8.(30)	1.(40)		X
F_3	5(40)		(70)	(20)	6
	X	X	6		

Table 3.9

	W_1	W_2	W_3	W_4	
F_1				7(10)	X
F_2		8(30)	1(40)		X
F_3	5(40)		6(70)		X
	X	X	X	X	

Step 6. The last allocation of amount x_{33} = 6 is obviously made in the cell (3, 3). This exhausts the availability from factory F_3 and requirement of warehouse W_3 simultaneously. So we cross-out third row and third column to get the final solution in Table 3.9.

The initial basic feasible solution and the corresponding transportation cost are displayed in table 3.10.

Table 3.10

	W_1	W_2	W_3	W_4
F_1				7
F_2		8	1	
F_3	5		6	7

The transportation cost is given by $z = 7 \times 10 + 8 \times 30 + 1 \times 40 + 5 \times 40 + 6 \times 70 + 7 \times 20 =$ Rs. 1110

Third method : The Column Minima Method

This method is similar to row-minima method except that we apply the concept of minimum cost on columns instead of tows. So, one can easily solve the above problem by column minima method also.

Fourth method : Lower Cost Entry Method (Matrix minima Method)

The initial basic feasible solution obtained by this method usually gives a lower beginning cost. In this method, first write the cost and requirements matrix (table 3.11).

Start with the lowest cost entry (8) in the cell (3, 2) and allocate as much as possible, i.e., $x_{32} = 8$. The next lowest cost (10) lies in the cell (1.4), so allocate $x_{14} = 7$. The next lowest cost (19) lies in the cell (1, 1), so make no allocation, because the amount available from factory F_1 was already used in the cell (1, 4). Next lowest cost entry is (20) in the cell (3, 4) where at the most it is possible to allocate $x_{34} = 7$ in order to complete the requirements of 7 units in column 4. Thereafter, next lowest cost is (30) in cells (2, 2) and (1, 2) so no allocation is possible, because the requirement of column 2 has already been exhausted. This way, required feasible solution is obtained (table 3.11).

Table 3.11

	W_1	W_2	W_3	W_4	
	(19)	(30)	(50)	7(10)	7
	2(70)	(30)	7(40)	(60)	9
	3(40)	8(8)	(70)	7(20)	18
Requirements	5	8	7	14	

The feasible solution results in lower transportation cost, i.e.,

$$2(70) + 3(40) + 8(8) + 8(40) + 7(10) + 7(20) = \text{Rs. } 814.$$

The cost is less by Rs. 201, i.e. Rs. (1015-814) as compared to the cost obtained by Northwest corner rule.

Fifth Method. Vogel's Approximation Method (Unit Cost Penalty Method)

Step 1. In lowest cost entry method, it is not possible to make an allocation to the cell (1, 1) which has the second lowest cost in the matrix. In is trivial that allocation should be made in at least one cell of each row and each column.

Table 3.12

	W_1	W_2	W_3	W_4	Available
F_1	(19)	(30)	(50)	(10)	7
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(8)	(70)	(20)	18
Requirement	5	8	7	14	

Step 2. Next enter the difference between the lowest and second lowest cost entries in each column beneath the corresponding column, and put the difference between the lowest and second lowest cost entries of each row to the right of that row. Such individual differences can be thought of a penalty for making allocation in second lowest cost entries instead of lowest cost entries in each row or column. For example, if we allocate one unit in the second lowest cost cell (3, 1) instead of cell (1, 1) with lowest unit cost (19), there will be a loss (penalty) of Rs. 21 per unit. In case, the lowest and second lowest costs in a row/column are equal the penalty will be taken zero.

Table 3.13

	W_1	W_2	W_3	W_4	Available	Penalties
F_1	(19)	(30)	(50)	(10)	7	(9)
F_2	(70)	(30)	(40)	(10)	19	(10)
F_3	(40)	8(8)	(70)	(20)	16	(12)
Requirement	5	8/0	7	14		
Penalties	(21)	(22)	(10)	(10)		

Step 3. Select the row or column for which the penalty is the largest, i.e., (22) (table 3.13), and allocate the maximum possible amount to the cell (3, 2) with the lowest cost (8) in the particular column (row) making $x_{32} = 8$. If there are more than one largest penalty rows (columns), select one of them arbitrarily.

Step 4. Cross-out that column (row) in which the requirement has been satisfied. In this example second column has been crossed-out. Then find the corresponding penalties correcting the amount available from factor F_3. Construct the first reduced penalty matrix (table 3.14).

Table 3.14

	W_1	W_3	W_4	Available	Penalties
F_1	5(19)	(50)	(10)	7	(9)
F_2	(70)	(40)	(60)	9	(20)
F_3	(40)	(70)	(20)	10(Note)	(20)
Requirement	5/0	7	14		
Penalties	(21)	(10)	(10)		

Step 5. Repeat steps 3 and 4 till all allocations have been made. Successive reduced penalty matrices are obtained. Since the largest penalty (21) is now associated with the cell (1, 1), so allocate $x_{11} = 5$. This allocation ($x_{11} = 5$) eliminates the column giving the second reduced matrix (table 3.15).

Table 3.15

	W_3	W_4	Available	Penalties
F_1	(50)	(10)	2(Note)	(40)
F_2	(40)	(60)	9	(20)
F_3	(70)	(20)	10/0	(50)
Requirement	7	14/4		
Penalties	(10)	(10)		

The largest penalty (50) is now associated with the cell (3, 4) therefore allocate $x_{34} = 10$. Eliminating the row 3, the third reduced penalty matrix table 3.16 is obtained.

Table 3.16

	W_3	W_4	Available	Penalties
F_1	(19)	2(10)	2/0	(40)
F_2	7(40)	2(60)	9/0	(20)
Requirement	7	4/0 (Note)		
Penalties	(10)	(50)		

Now, allocate according to the largest penalty (50) as $x_{14} = 2$ and remaining $x_{24} = 2$. Then allocate $x_{23} = 7$.

Step 6. Finally, construct table 3.17 for the required feasible solution.

Table 3.17

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	19
F_3		8(8)		10(20)	18
Requirement	5	8	7	14	

The total cost is :

$$5(19) + 8(8) + 2(10) + 2(60) + 10(20) + 7(40) = \text{Rs. } 779.$$

The cost is Rs. 35 less as compared to the cost obtained by Lowest Cost Entry Method.

In order to reduce large number of steps required to obtain the optimal solution, it is advisable to proceed with the initial feasible solution which is close to the optimal solution. Vogel's method often gives the better initial feasible solution to start with. Although Vogel's method takes more time as compared to other two methods, it reduces the time in reaching the optimal solution.

Summary of Methods for Initial BFS

The methods for obtaining an initial basic feasible solution to a transportation problem can be summarized as follows :

1. **North-West Corner Rule :**

Step 1. The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the transportation table. The maximum possible amount is allocated there. That is, $x_{11} = \min(a_1, b_1)$. This value of x_{11} is then entered in the cell (1, 1) of the transportation table.

Step 2.

(i) If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation of amount $x_{21} = \min(a_1, b_1 - x_{11})$ in the cell (2, 1).

(ii) If $b_1 < a_1$, move horizontally right-side to the second row and make the second allocation of amount $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).

(iii) If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude $x_{12} = \min\,(a_1 - a_1, b_2) = 0$ in the cell (1, 2) or $x_{21} = \min(x_2, b_1 - b_1) = 0$ in the cell (2, 1).

Step 3. Start from the new north-west corner of the transportation table and repeat steps 1 and 2 until all the requirements are satisfied.

2. **The Row Minima Method :**

Step 1. The smallest cost in the first row of the transportation table is determined. Let it be c_{ij}. Allocate as much as possible amount $x_{ij} = \min(a_1, b_j)$ in the cell $(1, j)$, so that either the capacity of origin 0_1 is exhausted, or the requirement at destination D_j is satisfied or both.

Step 2.

(i) If $x_{ij} = a_1$ so that the availability at origin 0_1 is completely exhausted, cross-out the first row of the table and move down to the second row.

(ii) If $x_{1j} = b_j$ so that the requirement at destination D_1 is satisfied, cross-out the j-th column and reconsider the first row with the remaining availability of origin 0_1.

(iii) If $x_{1j} = a_1 = b_j$, the origin capacity a_1 is completely exhausted and the requirement at destination D_j is also completely satisfied. The breaking choice is made arbitrarily. Cross-out the j-th column and make the second allocation $x_{1k} = 0$ in the cell $(1, k)$ with c_{1k} being the new minimum cost in the first row. Cross-out the first row and move down to the second row.

Step 3. Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied.

3. **The Column Minima Method**

Step 1. Determine the smallest cost in the first column of the transportation table. Let it be c_a. Allocate x_{i1} = min (a_1, b_1) in the cell (i, 1).

Step 2.

(i) If $x_{i1} = b_1$, cross-out the first column of the transportation table and move towards right to the second column.

(ii) If $x_{i1} = a_i$, cross-out the i-th row of the transportation table and reconsider the first column with the remaining demand.

(iii) If $x_{i1} = b_1 = a_i$, cross-out the i-th row and make the second allocation x_{k1} = 0 in the cell (k, 1) with c_{k1} being the new minimum cost in the first column. Cross-out the column and move towards right to the second column.

Step 3. Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied.

4. **Lowest Cost Entry Method (LCEM) or Matrix Minima Method**

Step 1. Determine the smallest cost in the cost matrix of the transportation table. Let it be (c_{ij}). Allocate x_{ij} = min (a_1, b_1) in the cell (i, j).

Step 2.

(i) If $x_{ij} = a_i$, cross-out the i-th row of the transportation table and decrease b_1 and a_i. Go to step 3.

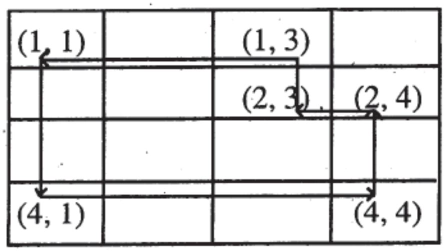
(ii) If $x_{ij} = b_j$, cross out the j-th column of the transportation table and decrease a_i by b_j. Go to step 3.

(iii) If $x_{ij} = a_i = b_j$, cross-out either the i-th row or j-th column but not both.

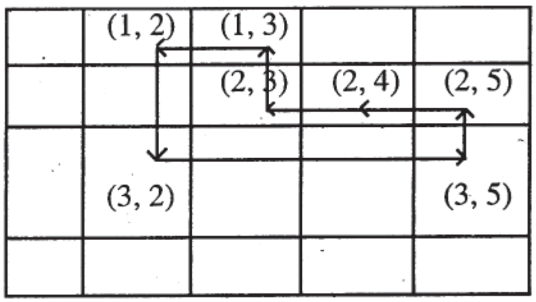
Step 3. Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost in not unique, make an arbitrary choice among the minima.

5. **Vogel's Approximation Method (VAM)**

Step 1. For each row of the transportation table identify the smallest and next-to-smallest cost. Determine the difference between them for each row. These are called



(i) Loop L



(ii) Non-loop L'

Fig. : 3.1

Theorem 1. *Every loop has an even number of cells.*

Proof. For any loop, we can always choose arbitrarily a starting point and a direction by an arrow mark (→). We consider a loop formed by n number of cells which are consecutively numbered from 1 to n. Now assume that cell 1 and 2 exist in the same column. Thus the step from cell 1 to cell 2 involves a row change. Obviously, step from cell 2 to cell 3 must involve a column change, from cell 3 to cell 4 a row change, and so on. In general, the step to cell k involves a row change, if and only if, k is even. Since the step to cell 2 involved a row change, the step from cell n to cell 1 must be a column change and the step from cell $n - 1$ to cell n a row change. Hence n will be even.

Set containing a loop. A set X of cells of a transportation table is said to contain a loop if the cells of X or a subset of X can be sequenced (ordered) so as to form a loop.

Theorem 2 (Linear Dependence and Loops). *Let X be a set of column vectors of the coefficient matrix of a transportation problem (T.P.). Then a necessary and sufficient condition for vectors in X to be linearly dependent is that the set of their corresponding cells in the transportation table contains a loop.*

Proof. Let us consider an m-origin, n-destination T.P. expressed in its matrix 'penalties'. Put them along side the transportation table by enclosing them in the parentheses against the respective rows. Similarly, compute these penalties for each column.

Step 2. Identify the row or column with the largest penalty among all the rows and columns. If a tie occurs, use any arbitrary tie breaking choice. Let the the largest penalty correspond to i-th row and let c_{ij} be the smallest cost in the i-th row. Allocate the largest possible amount $x_{ij} = \min(a_1, b_1)$ in the cell (i, j) and cross-out the i-th row or the j-th column in the usual manner.

Step 3. Again compute the column and row penalties for the reduced transportation table and then go to step 2. Repeat the procedure until all the requirements are satisfied.

3.5 Moving towards Optimal B.F.S.

Before proceeding to find the optimal solution one needs to know the following:

Loops in transportation table and their properties

Definition (Loop). In a transportation table, an ordered set of four or more cells is said to form a loop if,

(i) any two adjacent cells in the ordered set lie either in the same row in the same column; and

(ii) any three or more adjacent cells in the ordered set do not lie in the same row or in the same column.

The first cell of the set is considered to follow the last one in the set.

If we join the cells of a loop by horizontal and vertical lineup segments, we get a closed path satisfying the above conditions (i) and (ii). Let us denote the (i, j)th cell of the transportation table by (i, j). Then it can be observed from the diagrammatic illustration in Fig. 1 that the set $L = \{(1, 1), (4, 1), (4, 4), (2, 4), (2, 3), (1, 3)\}$ form a loop and on the other hand the set $L' = \{(3, 2), (3, 5), (2, 5), (2, 4), (2, 3), (1, 3), (1, 2)\}$ does not form a loop, because three cell entries (2, 3), (2, 4) and (2, 5) lie in the same row (second).

form :

Minimize $z = \boldsymbol{c'x : c, c \varepsilon R^{mn}}$, subject to the constraints : $\boldsymbol{Ax = b, x \geq 0, b \varepsilon R^{m+n}}$, where $b = (a_1, a_2, ..., a_m, b_1, b_2, ..., b_n)$, A is an $(m + n) \times mn$ real matrix containing the coefficients of constraints and cis the cost vector.

To prove that the condition is sufficient.

Let us assume that the cells associated with the vectors of X contain a loop

$L = \{(i, j), (i, k), (l, k), ..., (p, 0), (p, j)\}$.

If a_{ij} denotes the column vector of matrix A associated with the variable x_{ij} [the cell (i, j)], then it follows that $a_{ij} = e_i + e_{m+j}$, where $e_i, e_{m+j} \varepsilon R^{m+n}$ are unit vectors. Thus X includes the column vectors :

$a_{ij} = e_i + e_{m+j}, a_{ik} = e_i + e_{m+k}, a_{1k} = e_1 + e_{m+k}, a_{im} = e_i + e_{m+m}, ..., a_{po} = e_p + e_{m+0}$, and $a_{pj} = e_p + e_{m+j}$.

Hence by successive addition and subtraction, we get $a_{ij} - a_{ik} + a_{ik} - a_{1m} + ... + a_{p0} - a_{pj} = 0$

(since by theorem 1, a loop contains an even number of cells).

Therefore, this particular subset of X, and hence X itself, is a linearly dependent set.

To prove, the condition is necessary.

Let us assume that X is a linearly dependent set. Then, there must exist scalars λ_{ij} not all zero such that

$\Sigma\lambda_{ij}\, a_{ij} = 0$, where $a_{ij} \varepsilon X$.

For simplification, remove all those vectors from X for which $\lambda_{ij} = 0$.

Now we arbitrarily choose a vector from the remaining vectors in X. Let it be $a_{ij} = e_i + e_{m+j}$. We claim that X must contain at least one more vector whose second subscript is j. Suppose, to the contrary, it does not, then since $\lambda_{ij} = 0$, the $(m + j)$th component of the vector equation $\Sigma\lambda_{ij}\, a_{ij} = 0$ gives $\lambda_{ij} = 0$, which is a contradication. So X must contain at least one more vector with second subscript j.

Suppose that this vector is $a_{kj} = e_k + e_{m+j}$. By similar reasoning we conclude that there must be at least one more vector in X with the first subscript k, say, $a_{k1} = e_k + e_{m+1}$. By same argument once again, X must contain at least one vector with the second subscript 1. Let it be, say, $a_{il} = e_i + e_{m+1}$.

Thus we have determined four vectors in X, namely a_{ij}, a_{kj}, a_{k1} and a_{i1} whose corresponding cells, form a loop. Thus the proof is complete.

If the last vector is $a_{n1} = e_n + e_{m+1}$ instead of a_{i1}, then as explained just before there must exist at least one more vector with first subscript n. If it is a_{nj}, a loop is complete, if not, let it be $a_{n0} = e_n + e_{m+0}$. X must contain at least one more vector with second subscript 0. Now two cases will arise :

(1) The first subscript of newly discovered vector is one that has already been identified. In this case a loop has been completed.

(2) The first subscript of the newly discovered vector is also new. In this case, since the number of vectors in X is finite (by extending the above reasoning), we conclude that eventually a loop must be formed.

Corollary. A feasible solution to a transportation problem is basic if, and only if, the corresponding cells in the transportation table do not contain a loop.

This corollary provides us a method to verify whether the current feasible solution to the transportation problem is basic or not.

3.6 Moving towards Optimum Solution

After obtaining an initial basic feasible solution to a given transportation problem, the next question is 'how to arrive at the optimum solution'. The basic steps for reaching the optimum solution are the same as given for simple method, namely :

Step 1. Examination of initial basic feasible solution for non-degeneracy. If it is degenerate, some modification is required to make it non-degenerate).

Step 2.

(i) Determination of net-evaluations (cost-difference) for empty cells.

(ii) Optimality test of current solution.

Step 3. Selection of the entering variable, provided step 2(ii) indicates that the current solution can be improved.

Step 4. Selection of the leaving variable.

Step 5. Finally, repeating the steps 1 through 4 until an optiumum solution is obtained.

3.6.1 To examine the initial basic feasible solution for non-degeneracy

A basic feasible solution of an $m \times n$ transportation problem is said to be non-degenerate, if it has the following two properties :

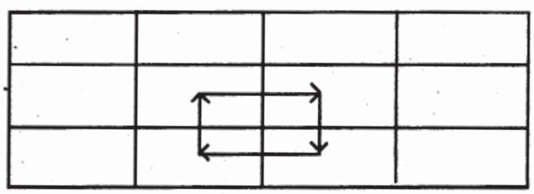
(1) Initial b.f.s. must contain exactly $m + n - 1$ number of individual allocations.

For example, in 3×4 transportation problem, the number of individual allocations in b.f.s. obtained by any one of the methods discussed so far is equal to 6 i.e., $3 + 4 - 1$, which can be easily verified from tables 3.3, 3.10, 3.11 and 3.17.

(2) These allocations must be in 'independent positions'.

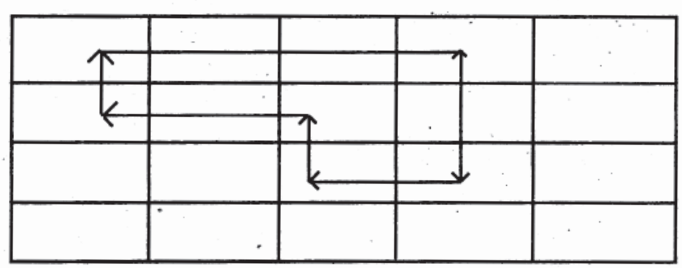
Independent positions of a set of allocations mean it is always impossible to form any closed loop through these allocations. Tables 3.18—3.20 show the non-independent, and 3.21 independent positions by '.'.

Table 3.18
Non-independent positions



Closed loop

Table 3.19
Non-independent positions



Closed loop

Table 3.20
Non-independent positions

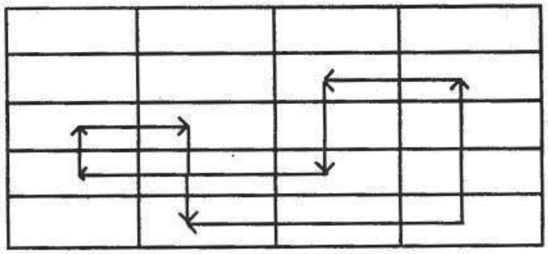


Table 3.21
Independent positions

In the above allocation patterns of different problems, the dotted lines constitute what are known as loops. A loop may or may not involve all allocations. It consists of (at least 4) horizontal and vertical lines with an allocation at each corner which, in turn, is a join of a horizontal and vertical lines. At this stage loop of table 3.20 should be particularly noted. Here two lines intersect each other at cell (4, 2) and do not simply join, therefore this is not to be regarded as a corner. Such allocations in which a loop can be formed are known as non-independent positions whereas those (of table 3.21) in which a loop cannot be formed are regarded as independent.

3.6.2 Determination of $z_j - c_j$ values—The U-V Method

Unlike the simplex method, the $z_j - c_j$ values (as defined in simplex method) for a transportation problem can be determined more easily by using the properties of the primal and dual problems.

Let us consider the following m-origin, n-destination transportation problem :

Determine x_{ij} so as to minimize = $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}(c_{ij})$, subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i \quad \text{or,} \quad a_i - \sum_{j=1}^{n} x_{ij} = 0, \text{ for } i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{or,} \quad b_j - \sum_{i=1}^{m} x_{ij} = 0, \text{ for } j = 1, 2, ..., n$$

and $\quad x_{ij} > 0$, for all i and j.

Let $u_1, u_2, ..., u_m$ and $v_1, v_2, ..., v_n$ be the dual variables associated with the above origin and destination constraints, respectively. Since the above primal T.P., has $m + n$ constraint equations in mn number of variables, so the dual of above problem will contain mn constraints in $m + n$ dual variables in the form :

$u_1 + v_j \leq c_{ij}$ and u_i, v_j are unrestricted for all i and j ($\because$ constraints in the primal are equations)

From duality, for any standard primal L.P.P. with basis B and associated cost vector c_B, the associated solution to its dual is given by $w_B = c_a B^{-1}$. Thus, if a_j be the j-th column of the primal coefficient matrix, then an expression for evaluating the $(z_j - c_j)$-values for minimization problem is given by

$$z_j - c_j = c'_B(B^{-1}a_j) - c_j = w'_B a_j - c_j \quad \text{for all } j.$$

But, in the case of transportation problem (which is the special case of L.P.P.), the dual solution can be represented by

$$(u', v') = (u_1, ..., u_m. v_1, ..., v_n)$$

and therefore the $z_j - c_j$ values are obtained by simply replacing $c_j \to c_{ij}$, $w_B \to (u', v')'$, $a_j \to a_{ij}$ in the above formula to get

$$z_{ij} - c_{ij} = (u', v')a_{ij} - c_{ij} = (u_1, ..., u_m, v_1, ..., v_n)\,[e_i + e_{m+j}] - c_{ij}$$

where a_{ij} is the column vector of the coefficient matrix associated with the variable x_{ij}. For simplicity, we shall denote $c_{ij} - z_{ij}$ by d_{ij}. d_{ij}'s are known as net evaluations.

Now, since the net evaluation must vanish for the basic variables it follows that $d_{ij} = c_{ij} - (u_i + v_j)$ for all non-basic cells (i, j) where u_i and v_j satisfy the relation $c_{rs} = u_r + v_s$ for all basic cells (r, s). Except for the degeneracy case, there are $m + n - 1$ dual equations in $m + n$ dual unknowns u and v, and solve uniquely for the remaining $m + n - 1$ variables. After this arbitrary assignment, say $u_1 = 0$, the rest of the values are obtained by simple addition and subtraction. Once we determine all the u_i and v_j, the net evaluations for all the non-basic cells are easily determined by the relation $d_{ij} = c_{ij} - (u_i + v_j)$.

Alternative method to determine net-evaluations

The necessary condition for optimality can also be established in the form of the following theorem.

Theorem 7. If we have a feasible solution consisting of $m + n - 1$ independent allocations, and if numbers u_i and v_j satisfying $c_{rs} = u_r + v_s$, for each occupied cell (r, s), then the evaluation d_{ij} corresponding to each empty cell (i, j), is given by $d_{ij} = c_{ij} - (u_i + v_j)$.

Proof. The transportation problem is to find $x_{ij} \geq 0$ in order to minimize

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}(c_{ij}) \quad \text{... (6)}$$

subject to the restrictions

$$\sum_{j=1}^{n} x_{ij} = a_j, j = 1, 2, 3, ..., m \quad \text{... (7)}$$

$$b_j - \sum_{i=1}^{m} x_{ij} = b_j, j = 1, 2, 3, ..., n \quad \text{... (8)}$$

and $x_{ij} \geq 0$, for all i and j.

The restrictions (7) and (8) may be written as

$$0 = a_i - \sum_{ji1}^{n} x_{ij}, i = 1, 2, 3, ..., m \quad \text{... (9)}$$

$$0 = b_j - \sum_{i=1}^{m} x_{ij}, j = 1, 2, 3, ..., n \quad \text{... (10)}$$

Any multiple of each ofthese restrictions [(9) and (10)] can be legally added to the objective function (6) to try to eliminate the basic variables. These multiples are denoted by $u_i (i = 1, 2, ..., m)$ and $v_j (j = 1, 2, ..., n)$, respectively. Thus,

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}(c_{ij}) + \sum_{i=1}^{m} u_i \left(a_i - \sum_{j=1}^{n} x_{ij} \right) + \sum_{j=1}^{n} v_j \left(b_j - \sum_{i=1}^{m} x_{ij} \right) \quad \text{... (11a)}$$

or, $$z = \sum_{i=1}^{m} \sum_{j=1}^{n} [c_{ij} - (u_i + v_j)] x_{ij} + \sum_{i=1}^{m} u_i a_i + \sum_{i=1}^{n} v_j b_j \quad \text{... (11b)}$$

The necessary condition for a coefficient to be zero is

$$c_{rs} = u_r + v_s \quad \text{... (12)}$$

for each basic variable x_{rs}, i.e., for each occupied cell (r, s). Since there are $m + n - 1$ number of equations of the form (12) in $(m + n)$ number ofunknown (u_i and v_j), so if assignment is made to an arbitrary value of one the u_i or v_j, then rest of the $(m + n - 1)$

unknowns can be easily solved algebraically. One reasonable and convenient rule, which will be adopted here, is to select the u_i which has the largest number of allocations in its row, and assign it the value of zero. And $c_{ij} = u_i + v_j$ immediately yields v_j for columns containing those allocations.

To prove the required result, first suppose that the empty cell (i, j) be connected to occupied cells by a closed loop (table 3.22).

First, allocate + 1 unit to the empty cell (i, j), and in order to balance the total requirement of warehouse W_j, and – 1 unit to occupied cell (r, j). Consequentyly, the total amount available from factor F_r will be balanced by adding + 1 unit to occupied cell (r, s), which in turn causes column W_s to become unbalanced. So balance the column W_s by adding – 1 unit to the occupied cell (i, s).

Table 3.22

	W_1	W_2	....	W_1	....	W_s		W_n	Available
F_1									a_1
F_2									a_j
.									.
.				$(c_{ij})+1$		$.(c_{is})-$			.
F_j						–			a_j
.									.
.				-1		$+1$			.
F_r				$.(c_{rj})$		(c_{rs})			a_r
.									.
.									.
F_m									a_m
Required	b_1	b_2	......	b_j	......	b_s	......	b_n	

This process will give the cost difference d_{ij} [called the empty cell evaluation for (i, j)] between the new solution and the original solution.

Thus, $$d_{ij} = c_{ij} - c_{rj} + c_{rs} - c_{is} \qquad ... (13)$$

Using the result (12) for all occupied cells such as (r, j), (r, s) and (i, s),

$$d_{ij} = c_{ij} - (u_r + v_j) + (u_r + v_s) - (u_i + v_s) = c_{ij} - (u_i + v_j). \quad \text{... (14)}$$

This proves the result for a loop of square shape connecting the empty cell (i, j) to occupied cells. In a similar fashion, one can generalize for a loop connecting the empty cell (i, j) to occupied cells.

3.6.3 The optimality test

If $d_{ij} \geq 0$ for all i, j then the b.f.s. under test must be optimal. But, if $d_{ij} < 0$ for at least one empty cell, then we can improve upon the solution. This way, it is possible to improve the b.f.s. successively till the optimal solution is obtained for which $d_{ij} \geq 0$ for each empty cell.

The optimality test for given b.f.s. of the transportation problem may be summarized as follows :

1. Start with a basic feasible solution consisting of $m + n - 1$ allocations in independent positions.

2. Determine a set of $m + n$ numbers $u_i (i = 1, 2, 3, ..., m)$ and $v_j (j = 1, 2, 3, ..., n)$ such that for each occupied cell (r, s) $c_{rs} = u_r + v_s$.

3. Calculate cell evaluations d_{ij} for each empty cell (i, j) by using the formula

$$d_{ij} = c_{ij} - (u_i + v_j).$$

4. Finally, examine the matrix of cell evaluations d_{ij} for negative entries and conclude that—

   (i) solution under test is optimal, if none is negative;

   (ii) alternative optimal solutions exist, if none is negative but any is zero;

   (iii) solution under test is not optimal, if any is negative, then further improvement is required by repeating the above process.

We now proceed to answer the question : how to improve the current b.f.s. if it is not optimal, i.e. if all $d_{ij} \leq 0$.

3.6.4 Selection of entering variable

Here our aim is to minimize the cost of transportation. So the current basic feasible solution will not be optimum so long as any of the net evaluation d_{ij} is negative. Thus if all d_{ij} are non-negative, the current solution is an optimum one, otherwise using simplex like criterion we select such variable x_{rs} to enter the basis for which the net evaluation $d_{rs} = \min_{i,j} \{d_{ij} < 0\}$.

3.6.5 Selection of leaving variable

Our next step will be to determine the basic variable to be removed and then to determine the new improved basic solution. The simplex like leaving criterion in the notations of transportation problem states that if the variable x_{rs} is selected to enter the basis, then the basic variable x_{Bi} corresponding to the minimum ratio $\min\left\{\frac{x_{Bi}}{y_{irs}}, y_{irs} > 0\right\}$, will leave the basis, where y_{ijk} corresponds to y_{ij} in simplex algorithm. However, due to special structure of the transportation problem the above criterion has been simplified to a great extent.

Theorem 3.8. Let $\{b_1, b_2, ..., b_{m+n-1}\}$ be a basis set for the column vectors of the coefficient matrix A of m-origin, n-destination transportation problem. In the representation of any non-basic vector a_{rs} as a linear combination :

$$a_{rs} = \sum_{i=1}^{m+n-1} y_{irs} b_i$$

of basis vectors, every scalar element (y_{irs}) is either -1 or $+1$.

Proof. Since $(a_{rs}, b_1, ..., b_{m+n-1}\}$ is a linearly dependent set, the set of the associated cells contains a loop. So the cell (r, s) must be in a loop. The vectors b_i's are, of course, some column vectors a_{ij}'s of A.

Suppose the set of associated cells contains the following loop :

$$L = \{(r, s).(r, t).(p, t).(p, q), ..., (u, v).(u, s)\}.$$

where $a_{rt}, ..., a_{us}$ are the given basis vectors $(\leq m + n - 1)$.

Now, since $a_{ij} = e_i + e_{m+j}$ for all i and j, and because the number of cells in a loop is always even, we have

$$a_{rs} - a_{rt} + a_{pt} - a_{pq} + ... + a_{uv} - a_{us} = 0$$

which yields, $$a_{rs} = a_{rt} - a_{pt} + a_{pq} - ... - a_{uv} + a_{us}.$$

This is the unique representations of a_{rs} as a linear combination of basis vectors; and hence the y_{irs} elements associated with the basis vectors in the above representation are $+1, -1, ..., -1$ and $+1$.

This completes the proof of the theorem.

Thus if x_{rs} is the entering variable, the basic variable x_{Bt} will leave if x_{Bt} = min $\{x_{Bi}\}$, since positive y_{irs} is + 1 for all basic variables.

3.6.6 Determination of new (improved) basic feasible solution

After the entering variable x_{rs} and leaving variable x_{Bt} are determined, all that remains to determine is 'the new (improved) basic solution'. The usual transformation formulae for obtaining the new basic solution, in the simplex transportation notation, are given by

$$\hat{x}_{Bt} = \frac{x_{Bt}}{y_{trs}}, y_{trs} > 0 \text{ and } \hat{x}_{Bt} = x_{Bt} - \frac{x_{Bt}}{y_{trs}}, y_{irs} \text{ for all } i \neq t.$$

Since $y_{trs} = +1$, $y_{irs} = \pm 1$ for basic variables, and $y_{irs} = 0$ for non-basic variables, the above transformations get simplified to

$$\hat{x}_{Bt} = x_{Bt},$$

$$\hat{x}_{Bt} = x_{Bt} - y_{irs} x_{Bt} = x_{Bi} \pm x_{Bt} \text{ for all } x_{Bt} \;\varepsilon\; X, i \leq t$$

and $\hat{x}_{Bt} = x_{Bt}$, for al $x_{Bi} \notin X$,

where X is the set of those basic variables whose corresponding cells are included in the loop as identified in the preceding theorem.

In practice, however, this improvement procedure is quite simple. The working-rule is outlined in the following steps :

3.6.7 Working rule to obtain leaving variable and improved basic feasible solution :

Step 1. After identifying the entering variable x_{rs}, describe a loop which starts and ends at the non-basic cell (r, s) connecting only the basic cells. Such a closed path exists and is unique for any non-degenearate basic solutions.

Step 2. The amount (say θ) to be allocated to the entering variable is interchangeably subtracted from and added to the successive end points of the closed loop so that the supply and demand constraints always remain satisfied.

Step 3. Then the minimum value of θ, which will render non-negative values for all the basic variables in the new solution, is obtained. This consequently, determines the leaving variable.

3.7 Transportation Algorithm for Minimization Problem

The transportation algorithm for minimization problem can be summarized in the following steps.

The Algorithm :

Step 1. First construct a transportation table entering the origin capacities a_i, the destination requirements b_j and the costs c_{ij}.

Step 2. Find an initial basic feasible solution by Vogel's method or by any of the given methods. Enter the solution in the basic cells.

Step 3. For all the basic variable x_{ij}, solve the system of equations $u_i + v_j = c_{ij}$, for all i, j for which cell (i, j) is in the basis, starting initially with some $u_i = 0$ and entering successively the values of u_i and v_j on the transportation table as shown in table 3.24.

Step 4. Compute the cost differences $d_{ij} = c_{ij} - (u_i + v_j)$ for all the non-basic cells andenter them in the upper right corners of the corresponding cells.

Step 5. Apply optimality test by examining the sign of each d_{ij} :

(i) If all $d_{ij} \geq 0$, the current basic feasible solution is an optimum one.

(ii) If at least one $d_{ij} < 0$ (negative), select the variable x_{rs} (having the most negative d_{rs}) to enter the basis.

Step 6. Let the variable x_{rs} enter the basis. Allocate an unknown quantity say θ, to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount θ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step 7. Assign the largest possible value to θ in such a way that the value of at least one basic variable becomes zero and other basic variables remain non-negative ($\geq$ 0). The basic cell whose allocation has been made zero will leave the basis.

Step 8. Now, return to step 3 and then repeat the process until an optimum basic feasible solution is obtained.

The above interactive procedure determines an optimum solution in a finite number of steps. This method is called MODI METHOD and can be easily remembered.

Computational demonstration of optimality test

Example 2. Obtain an initial basic feasible solution to the transportation problem of Example

1. (a) Is this solution an optimal solution? If not, obtain the optimal solution.

(b) If a company is spending Rs. 1000 on transportation of its units to four warehouses from three factories, what can be the maximum saving by optimal scheduling?

Table 3.23

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	9
F_3		8(8)		10(20)	18
	5	8	7	14	

Solution : (a) Computational demonstration for optimality is performed by taking the initial basic feasible solution of Example 1 with $m + n - 1$ allocation is independent positions with transportation cost of Rs. 779 obtained (by Vogel's Method). This initial basic feasible solution is given in table 3.23.

Step 1. The initial b.f.s. has $m + n - 1$ allocations, that is, $3 + 4 - 1 = 6$ allocation in independent positions. Therefore, condition (1) of optimality test [in sec. 3.6.3] is satisfied.

Step 2. Since $u_i (i = 1, 2, 3)$ and $v_j (j = 1, 2, 3, 4)$ are to be determined by means of unit cost in the respective occupied cells only, assign a u-value of any particular amount (conveniently zero) to any particular row (convenient rule is to select the u_i which has the largest number of allocations in its row). Since all rows contain the same number of allocations, take any of the u_i (say u_3) equal to zero.

Table 3.24

	W_1	W_2	W_3	W_4	u_1
	(19)			.(10)	– 10
			(40)	.(60)	40
		(8)		.(20)	0
v_j	29	8	0	20	

When $u_3 = 0$, $v_4 = 20$ (since $c_{34} = u_3 + v_4$; $c_{34} = 20$). Similarly, $c_{32} = u_3 + v_2$ or $8 = 0 + v_2$ or $v_2 = 8$. Again, $c_{14} = u_1 + v_4$ or $10 = u_1 + 20$ (since $c_{14} = 10$), then $u_1 = -10$. In the same way : $60 = 20 + u_2$, which gives $u_2 = 40$, $19 = u_1 + v_1$ or $19 = -10 + v_1$, which gives $v_1 = 29$, $40 = u_2 + v_3$ or $40 = 40 + v_3$, which gives $v_3 = 0$. This completes the set of $u_i (i = 1, 2, 3)$ and $v_j (j = 1, 2, 3, 4)$ as shown in table 3.24.

Step 3. To compute the matrix of cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells, it is convenient to write a matrix $[c_{ij}]$ for empty cells and the matrix of numbers $[u_i + v_j]$ for empty cells only, then subtract the later matrix from the former one:

Table 3.25 (from table 3.3)
Matrix $[c_{ij}]$ for empty cells

.	(30)	(50)	.
(70)	(30)	.	.
(40)	.	(70)	.

Table 3.26
Matrixf $[u_i + v_j]$ for empty cells

	– 2	– 10	
69	48		
29		0	

Now, subtracting the matrix $[u_i + v_j]$ from the matrix $[c_{ij}]$, i.e., (table 3.25—table 3.26), the following matrix $[c_{ij} - (u_i + v_j)]$ of cell evaluation is obtained.

Table 3.27

.	32	60	.
1	$-18\surd$	.	.
11	.	70	.

Table 3.27 gives the empty cell evaluations : $d_{12} = 32$, $d_{13} = 60$, $d_{21} = 1$, $d_{22} = -18$, $d_{31} = 11$ and $d_{33} = 70$. The largest negative cell evaluation (marked $\surd$) is $d_{22} = -18$. So allocate (say, θ) to cell (2, 2) as much as possible; followed by alternately subtracting and adding the amount of this allocation to other corners of the loop in order to restore feasibility (non-negativity of allocation). For this purpose, the initial basic feasible solution can be read from table 3.28. It is easily seen by the following rule that at the most $\theta = 2$ units can be allocated from cell (2, 4) to cell (2, 2) still satisfying the row and column total and non-negativity restrictions on the allocations.

Table 3.28

	5 19			2 (10)	Available 7
	$+\theta$	7 (40)		$2-\theta$ (60)	9
	$8-\theta$ (8)			$10+\theta$ (20)	18
Required	5	8	7	14	

A rule to determine θ : Reallocation is done by transferring the maximum possible amount θ in the marked ($\surd$) cell. The value of θ, in general, is obtained by equating to zero the minimum of the allocations coptaining $-\theta$ (not $+\theta$) at the min $[8-\theta, 2-\theta] = 0$ or $2-\theta = 0$ or $\theta = 2$ units. Thus improved basic feasible solution is given in table 3.29.

Table 3.29

					Available
	5(19)			2(10)	7
		2(30)	7(40)		9
		6(8)		12(20)	18
Required	5	8	7	14	

The cost for this solution becomes

$$= 5(19) + 2(10) + 2(30) + 7(40) + 6(8) + 12(20) = \text{Rs. } 743.$$

The cost of Rs. 743 is Rs. 36 less than Rs. 779.

Step 4. Test this improved solution (table 3.29) for optimality by repeating steps 1, 2 and 3. In each step, following matrices are obtained.

Table 3.30

Matrix $[c_{ij}]$ for empty cells

.	(30)	(50)	.
(70)	.	.	(60)
(40)	.	(70)	.

Table 3.31

Matrix $[u_i + v_j]$ for empty cells

					U_i
	(19)			(10)	– 10
		(30)	(40)		22
		(8)		(20)	0
$v_j \rightarrow$	29	8	18	20	

Table 3.32

Matrix $[u_i + v_j]$ for empty cells

.	−2	8	.
41	.	.	42
29	.	18	.

Table 3.33

Matrix $[u_i + v_j]$ for empty cells

.	32	42	.
29	.	.	18
11	.	52	.

Since none of the cell evaluations is negative, i.e., $d_{12} = 32$, $d_{13} = 42$, $d_{21} = 29$, $d_{24} = 18$, $d_{31} = 11$ and $d_{33} = 52$, the solution given in table 3.29 is optimal with minimum cost of Rs. 743.

(b) Maximum saving = Rs. 1000 – Rs. 743 = Rs. 257.

3.8 Unbalanced Transportation Problems

So far we have discussed the balanced type of transportation problems where the total destination requirement equals the total origin capacity (i.e., $\Sigma a_i = \Sigma b_j$). But, sometimes in practical situations, the demand may be more than the availability or vice versa (i.e., $\Sigma a_i \neq \Sigma b_j$).

If in a transportation problem, the sum of all available quantities is not equal to the sum of requirements, that is, $a\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, the problem is called an unbalanced transportation problem.

3.8.1 To modify unbalanced T.P. to balanced type

An unbalanced T.P. may occur in two different forms (i) excess of availability, (ii) shortage in availability.

We now discuss these two cases by considering our usual m-origin, n-destination T.P. with the condition that $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$.

Case 1. (Excess availability, $\Sigma a_i \geq \Sigma b_j$).

The general T.P. may be stated as follows :

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}c_{ij}, \text{ subject to the constraints}$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i, \qquad i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, ..., n$$

$$\text{and} \qquad x_{ij} \geq 0, \qquad i = 1, 2, ..., m; j = 1, 2, ..., n.$$

The problem will possess a feasible solution if $\Sigma a_1 \geq \Sigma b_j$. In the first constraint, the introduction of slack variable $x_{i,n+1}$ $(i = 1, 2, ..., m)$ gives

$$\sum_{j=1}^{n} x_{ij} + x_{i,n+1} = a_i, \qquad i = 1, 2, ..., m$$

$$\text{or,} \quad \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} + x_{i,n+1}\right) = \sum_{i=1}^{m} a_i \quad \text{or,} \quad \sum_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij}\right) + \sum_{i=1}^{m} x_{i,n+1} = \sum_{i=1}^{m} a_i$$

$$\text{or,} \quad \sum_{i=1}^{m} x_{i,n+1} = \sum_{j=1}^{m} a_i - \sum_{j=1}^{n} b_i = \text{excess of availability} \left[\because \sum_{i=1}^{m} x_{ij} = b_j\right]$$

If this excess availability is denoted by b_{n+1}, the modified general T.P. can be reformulated as :

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n+1} x_{ij}(c_{ij}), \text{ subject to the constraints :}$$

$$\sum_{j=1}^{n} x_{ij} + x_{i,n+1} = a_i, \qquad i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, ..., n+1$$

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j,$$

where $c_{i,n+1} = 0$ for $i = 1, 2, ..., m$ and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n+1} b_j$.

This is clearly the balanced T.P. and thus can be easily solved by transportation algorithm.

Working rule : Whenever $\Sigma a_1 \geq \Sigma b_j$, we introduce a dummy destination-column in the transportation table. The unit transportation costs to this dummy destination are all set equal to zero. The requirement at this dummy destination is assumed to be equal to the difference $\Sigma a_i \geq \Sigma b_j$.

Case 2. (Shortage in availability, i.e., $\Sigma a_i \geq \Sigma b_j$)

In this case, the general T.P. becomes :

Minimize $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}(c_{ij})$,, subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, ..., m$$

$$\sum_{i=1}^{m} x_{ij} \leq b_j, \qquad j = 1, ..., n$$

$$x_{ij} \geq 0, \qquad i = 1, ..., m; j = 1, ..., n.$$

Now, introducing the slack variable x_{m+1}, ($j = 1, ..., n$) in the second constraint, we get

$$\sum_{i=1}^{m} x_{ij} + x_{i,n+1} = b_j, \qquad j = 1, ..., n$$

or, $$\sum_{j=1}^{m}\left[\sum_{j=1}^{m} x_{ij} + x_{m,1,j}\right] = \sum_{j=1}^{n} b_i \text{ or, } \sum_{i=1}^{m}\left[\because \sum_{j=1}^{m} x_{ij}\right] + \sum_{j=1}^{n} x_{m,1,j} = \sum_{j=1}^{n} b_i$$

or, $$\sum_{j=1}^{n} x_{m,1,j} = \sum_{j=1}^{n} - \sum_{i=1}^{n} a_i = \text{shortage in availability } a_{m+1}, \text{ say.}$$

Thus the modified general T.P. in this case becomes :

Minimize $z = \sum_{i=1}^{m+1} \sum_{j=1}^{n} x_{ij}c_{ij}$,, subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, ..., m+1$$

$$\sum_{i=1}^{m} x_{ij} + x_{m+1,j} \leq b_j, \qquad j = 1, ..., n$$

$$x_{ij} \geq 0, \qquad i = 1, ..., m + 1; j = 1, ..., n.$$

where $c_{i,,n,\,1} = 0$ for $j = 1, ..., n$ and $\sum_{i=1}^{m-1} a_i = \sum_{j=1}^{n+1} b_j$.

Working rule : Whenever $\Sigma a_i \geq \Sigma b_j$, introduce a dummy source in the transportation table. The cost of transportation from this dummy source to any destination are all set equal to zero. The availability at this dummy source is assumed to be equal to the difference $(\Sigma a_1 \geq \Sigma b_j)$.

Thus, an unbalanced transportation problem can be modified to balanced problem by simply introducing a fictitious sink in the first case and a fictitious source in the second. The inflow from the source to a fictitious sink represents the surplus at the source. Similarly, the flow from the fictitious source to a sink represents the unfilled demand at that sink. For convenience, costs of transporting a unit item from fictitious sources or to fictitious sinks has the case may be are assumed to be zero. The resulting problem then becomes balanced one and can be solved by the same procedure as explained earlier. The method for dealing with such type of problems will be clear in the example below :

Example 3. A steel company has three open hearth furnances and five rolling mills. Transportation cost (rupees per quintal) for shipping steel from furnaces to rolling mills are shown in the following table :

Table 3.40

Mills

		M_1	M_2	M_3	M_4	M_5	Capacities (in quintals)
Furnaces	F_1	4	2	2	2	6	8
	F_2	5	4	5	2	1	12
	F_3	6	5	4	7	3	14
	Requirement (in quintals)						

What is the optimal shipping schedule?

Solution : Since the total requirements of mills are 30 quintals and the total capacities of all furnaces are 34 quintals, the problem is of unbalanced type. Therefore, the problem can be modified as follows.

Step 1. (Modifying the given problem to balanced type)

Since the capacities are four quintals more than the total requirements, consider a fictitious mill requiring four quintals of steel. Thus the modified (balanced) transportation cost matrix becomes :

Table 3.40

Mills

	M_1	M_2	M_3	M_4	M_5	M_t	Capacities (in quintals)
F_1	4	2	3	2	6	0	8
F_2	5	4	5	2	1	0	12
F_3	6	5	4	7	3	0	14
Requirement	4	4	6	8	8	4	34

Step 2. (To find the initial solution).

Applying the Vogel's method in the usual manner, the initial solution is obtained as given below :

Table 3.41

	M_1	M_2	M_3	M_4	M_5	M_t
F_1		4(2)		4(2)		
F_2				4(2)	8(1)	
F_3	4(6)		6(4)			4(0)

This gives the transportation cost = 4(2) + 4(2) + 4(2) + 4(2) + 8(1) + 4(6) + 6(4) + 4(0) = Rs. 80.

Step 3. (To test the initial solution for optimality)

Since the total number of allocations is 7 (instead of 6 + 3 − 1 = 8), this is a degenerate basic feasible solution. Therefore, allocate an infinitesimal quantity Δ to empty cell (1, 1). Then, proceeding in the usual manner, following tables for testing the optimality of the solution are obtained.

Table 3.42

							u_i
	Δ(4)	.(2)	0	.(2)			0
				.(2)	.(1)		2
	.(6)		.(4)			.(0)	
v_j	4	2	2	2	1	– 2	

Table 3.43

$(u_i + v_j)$ for empty cells

.	.	2	.	1	– 2	0
4	2	2	.	.	– 2	0
.	4	.	4	3	.	2
4	2	2	2	1	– 2	

Table 3.44

$D_{ij} = c_{ij} - (u_i + v_j)$ for empty cells

.	.	1	.	5	8
1	2	3	.	.	2
.	1	.	3	.	.

Since all $d_{ij} = c_i - (u_i + v_j)$ for empty cells are non-negative, the solution under test is optimal. Further, 0 in the cell (3, 5) indicates that alternative solution will also exist.

Example 4. A company has three plants at locations *A*, *B* and *C* which supply to warehouses located at *D*, *E*, *F*, *G* and *H*. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in Rs.) are given below :

To

		D	E	F	G	H
From	A	5	8	6	6	3
	B	4	7	7	6	5
	C	8	4	8	6	4

Determine an optimum distribution for the company in order to minimize the total transportation cost.

Solution : In this problem, the total warehouse requirements (= 2500 units) is greater than the total plant capacity (= 2200 units). Therefore, the problem is of unbalanced type. So introduce a dummy plant *P* having all transportation costs equal to zero and having the plant availability equal to (2500 – 2200) = 300 units. The modified transportation table is thus obtained as below :

To

		D	E	F	G	H	Plant capacity
From	A	5	8	6	6	3	800
	B	4	7	7	6	5	500
	C	8	4	6	6	4	900
	P	0	0	0	0	0	300
	Requirements	400	400	500	400	800	

Using Vogel's Method the following initial b.f.s. obtained :

	D	E	F	G	H	Plant capacity
A			500(6)		300 (3)	800
B	400(4)			100(6)	Δ(5)	$500 + \Delta = 500$
C		400(4)			500(4)	900
P				300(0)		300
	400	400	500	400	$800 + \Delta = 800$	

Since the number of basic cell allocations (= 7) is less than $m + n - 1$ (= 8), the solution is degenerate. To make the number of allocations equal to 8 introduce a negligible small positive quantity Δ in the independent cell (2, 5). Now test the current solution for optimality.

Starting table

					u_i
+ (5) 2	+ (8) 3	500-θ (6)	+ (5) 4	300+θ (3)	− 2
400 (4)	+ (7) 8	− 1 (7) 8	100+θ (6)	Δ−θ (5)	0
+ (8) 3	400 (4) (6)	− 1 7 (6)	+ 5 (4)	500	− 1
+ (0) − 2	400 (0) − 1	− 2 +θ (0) 2	100-θ (0) 5	+ (0) − 1	6
v_i 4	5	8	6	5	

Here θ = min [500, Δ, 300] = Δ. So enter the non-basic cell (4, 3) and leave the basic cell (2, 5).

Since the number of basic cell allocations (= 7) is less than $m + n - 1$ (= 8), the solution is degenerate. To make the number of allocations equal to 8 introduce a negligible small positive quantity Δ in the independent cell (2, 5). Now test the current solution for optimality.

Starting table

					u_1
+ (5) 2	+ (8) 3	500-θ (6)	+ (5) 4	300+θ (3)	− 2
400 (4)	+ (7) 8	− 1 (7) 8	100+θ (6)	Δ−θ (5)	0
+ (8) 3	400 (4) (6)	− 1 7 (6)	+ 5 (4)	500	− 1
+ (0) − 2	400 (0) − 1	− 2 +θ (0) 2	100-θ (0) 5	+ (0) − 1	6
v_i 4	5	8	6	5	

Here θ = min [500, Δ, 300] = Δ. So enter the non-basic cell (4, 3) and leave the basic cell (2, 5).

Second Iteration Table. Vacate the cell (4, 4) and occupy the cell (3, 4).

						u_1
	+ (5) 3	+ (8) 3	200-θ (6)	+ (6) 5	600+θ (3)	– 1
	400 (4)	+ (7) 4	0 (7) 7	100 (6)	+ (5) 4	0
	+ (8) 4	400 (4)	– 1 θ (6) 7	300 (6)	200-θ (4)	0
	+ (0) – 3	+ (0) – 3	300 (0)	+ (0) – 1	+ (0) – 3	– 7
v_i	4	4	7	6	4	

Here min [200-θ, 200-θ] = 0 ⇒ θ = 200.

So introduce the cell (3, 3) and drop the cell (1, 3) or (3, 5) in the next iteration.

Third Iteration Table. Vacate the cell (1, 3) or (3, 5) and occupy the cell (3, 3).

Optimum Table

						u_i
	+ (5) 4	+ (8) 4	0 (6)	0 (6) 6	800 (3)	0
	400 (4)	+ (7) 4	+ (7) 6	100 (6)	+ (5) 3	0
	+ (8) 4	400 (4)	200 (6)	300 (6)	+ (4) 3	0
	+ (0) −2	+ (0) −2	300 (0)	0 (0) 0	+ (0) −2	−6
v_i	4	4	6	6	3	

Since all the net evaluations are non-negative, the optimum solution is :

$$X_{13} = 0, x_{15} = 800, x_{21} = 400, x_{24} = 100, x_{32} = 400, x_{33} = 200, x_{34} = 300, x_{43} = 300.$$

The optimum transportation cost is given by

$$z = 0(6) + 800(3) + 400(4) + 100(6) + 400(4) + 200(6) + 300(6) + 300(0)$$

$$= \text{Rs. } 9200.$$

3.9 Questions

1. What is a transportation problem? Write a linear programing model of a transportation problem.

2. Find an initial basic feasible solution to the following transportation problem using

(i) Northwest Corner rule
(ii) Column minima method
(iii) Row minima method
(iv) Matrix minima
(v) Vogel's approximation method

		To			
		1	2	3	Supply
From	1	2	7	4	5
	2	3	3	1	8
	3	5	4	7	7
	4	1	6	2	14
	Demand	2	9	18	

3. A manufacturing company has three factories F_1, F_2 and F_3 with monthly manufacturing capacities of 7000, 4000 and 10,000 units of a product. The product is to be supplied to seven stores. The manufacturing costs in these factories are slightly different but the important factor is the shipping cost from a factory to a store. The following table represents the factory capacities, the store requirements and the unit transportation costs.

		Stores							Factory
		S_1	S_2	S_3	S_4	S_5	S_6	S_7	Capacity
Factory	F_1	5	6	4	3	7	5	4	7000
	F_2	9	4	3	4	3	2	1	4000
	F_3	8	4	2	5	4	8	1	10,000
Store Demand		1500	2000	4500	4000	2500	3500	3000	

Find the optimal transportation plan so as to minimize the total transportation cost.

3. A company has factories at four different places (1, 2, 3 and 4) which supply items to warehouses *A, B, C, D* and *E*. Monthly factory capacities are 200, 175, 150 and 325, respectively, Monthly warehouse requirements are 110, 90, 120, 230 and 160 respectively. Unit shipping costs (in rupees) are given in the following table :

		To				
		A	B	C	D	E
From	1	13	–	31	8	20
	2	14	9	17	26	10
	3	25	11	12	17	15
	4	10	21	13	–	17

Shipment from 1 to *B* and from 4 to *D* are not possible. Determine the optimum distribution plan to minimize the shipping cost.

4. A company has plants at *A, B* and *C* which have capacities to produce 300 Kg., 200 Kg. and 500 Kg. respectively of a particular chemical per day. The production costs per Kg. in these plants are 70, Rs. 60 and Rs. 66 respectively. Four consumers have placed orders for the product on the following basis :

Consumer	Kg. required per day	Price offered (Rs./Kg)
I	400	100
II	250	100
III	350	102
IV	150	103

Shipping costs (in rupees per Kg.) from plants to consumers are given in the table below :

		To			
		I	II	III	IV
From	A	3	5	4	6
	B	8	11	9	12
	C	4	6	2	8

Find the optimal schedule for the situation.

Unit 4 ❑ Assignment Problems

Structure

4.1 Introduction

Assignment problems are also allocation problems. A typical allocation problem is as follows :-

Suppose there are n jobs to be assigned to n persons. Each person can perform only one job at a time with varying degree of efficiency. If the cost (payment) c_{ij} to be made to i-th person to perform the j-th job be given for all i and j, then the problem is to find how the n jobs should be allocated to the n persons so as to minimize the total cost. The assignment problem can be stated in form of a $n \times n$ cost matrix (c_{ij}) as given below:

Table 4.1

		Jobs					
Persons	1	c_{11}	c_{12}	......	c_{1j}	......	c_{1n}
	2	c_{21}	c_{22}	......	c_{2j}	......	c_{2n}
		...					
	i	c_{i1}	c_{i2}	......	c_{ij}	......	c_{in}
		...					
	n	c_{n1}	c_{n2}	......	c_{nj}	......	c_{nn}

4.2 Mathematical Formulation of Assignment Problem

Mathematically, an assignment problem can be stated as follows :

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to } x_{ij} = \begin{cases} 1 \text{ if the } i\text{-th person is assigned the } j\text{-th job} \\ 0 \text{ otherwise} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ (i.e., one job is done by } i\text{-th person)} \qquad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{m} x_{ij} = 1 \text{ (i.e., } j\text{-th job is done by one person)} \qquad j = 1, 2, \ldots, m,$$

Clearly, an assignment problem is a particular case of a transportation problem with $x_{ij} = 0$ or 1 for all i and j.

4.3 Hangarian Method for Assignment Problem

The solution technique of the assignment problem can be easily explained by the following example.

Example 1. A department has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How should the tasks be allocated to each person so at to minimize the total manhours?

Table 4.2

Subordinates

		I	II	III	IV
	A	8	26	17	11
	B	13	28	4	26
Tasks	*C*	38	19	18	15
	D	19	26	24	10

Solution : To understand the problem initially, step by step solution procedure is necessary.

Step 1. Subtracting the smallest element in each row from every element of that row, we get the reduced matrix (table 4.3).

Table 4.3

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Step 2. Next subtract the smallest element in each column from every element of that column to get the second reduced matrix (table 4.4).

Table 4.4

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Step 3. Now, test whether it is possible to make an assignment using only zeros.

If it is possible, the assignment must be optimal by theorem 2 of section 4.3. Zero assignment is possible in table 4.4 as follows :

(a) Starting with row 1 of the matrixf (table 4.4), examine the rows one by one until a row containing exactly single zero elements is found. Then an experimental assignment (denoted by*) is marked to that cell. Now cross all other zeros in the column in which the assignment has been made. This eliminates the possibility of marking further assignments in that column. The illustration of this procedure is shown in table 4.5.

Table 4.5

	I	II	III	IV
A	0*	14	9	3
B	9	20	0*	22
C	23	0	3	0
D	9	12	14	0*

(b) When the set of rows has been completely examined, an identical procedure is applied successively to columns. Starting with column 1, examine all columns until a column containing exactly one zero is found. Then make an experimental assignment in that position and cross other zeros in the row in which the assignment has been made.

Table 4.6

	I	II	III	IV
A	0*	14	9	3
B	9	20	0*	22
C	23	0*	3	⊗
D	9	12	14	0*

Continue these successive operations on rows and columns until all zeros have been either assigned or crossed out. At this stage, re-examine rows. It is found that no additional assignments are possible. Thus, the complete 'zero assignment' is given by $A \to$ I, $B \to$ III, $C \to$ II, $D \to$ IV as mentioned in table 4.6. According to theorem 1, this assignment is also optimal for the original matrix (table 4.2). Now compute the minimum total man-hours as follows :

Optimal assignment	*B*-I	*B*-II	*C*-II	*D*-IV	
Man-hour	8	4	19	10	(Total 41 hours)

Now the question arises : what would be further steps if the complete optimal assignment after applying step 3 is not obtained? Such difficulty will arise whenever all zeros of any row or column are crossed-out. Following example will make the procedure clear.

Example 2. A car hire company has one car at each of five depots *a, b, c, d* and *e*. A customer requires a car in each town, namely *A, B, C, D* and *E*. Distances (in kms) between depots (origins) and towns (destinations) are given in the following distance matrix :

Table 4.7

	a	*b*	*c*	*d*	*e*
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How should the cars be assigned to customers so as to minimize the distance travelled?

Solution : Applying steps 1 and 2 as explained in example 1 we get the table 4.8.

Table 4.8

30	0*	35	30	15
15	⊗	0*	10	⊗
30	⊗	35	30	20
0*	⊗	20	⊗	5
20	⊗	25	15	15

Now, column 1 has a single zero in row 5. Make an assignment by putting '*' and cross the other zero from the row which are not yet crossed. Column 3 has a single zero in row 2, make an assignment and delete the other zeros which are uncrossed.

It is observed that there are no remaining zeros; and row 3, row 5, column 4, and column 5 each has no assignment. Therefore, desired solution cannot be obtained at this stage, we now, proceed to the following important steps.

Step 4. Draw the minimum number of horizontal and vertical lines necessary to convert all zeros at least once. It should, however, be observed that (in all $n \times n$ matrices) less than n lines will cover zeros only when there is no solution among them. Conversely, if minimum number of lines is n, there is a solution. Following systematic procedure may help us to drow the minimum set of lines.

1. For simplicity, first make the table 4.8 again and name it as table 4.9.

Table 4.9

		$\surd L_1$				
	30	0	35	30	15	√
	~~15~~	~~⊗~~	~~0~~	~~10~~	~~⊗~~	L_2
	30	⊗	35	30	20	√
	0	⊗	20	⊗	5	L_3
	20	⊗	25	15	15	√

2. Mark (√) row 3 and row 5 as they are having no assignments and column 2 as having zeros in the marked rows 3 and 5.

3. Mark (√) row 1 because this row contains assignment in the marked column 2. No further rows or columns will be required to mark during this procedure.

4. Now start drawing required lines as follows :

   First draw line (L_1) through marked column 2. Then draw lines (L_2 and L_3) through left uncovered, the required lines will be (L_1, L_2 and L_3).

Step 5. In this step :

(i) first select the smallest element, say x, among all uncovered elements of the table 4.9 [as a result of step 4] and

(ii) then subtract this value x from all values in the matrix not covered by lines and add x to all those values that lie at the intersection of any two of the lines L_1, L_2 and L_3 (Justification of this rule is given on the next page).

After applying these two rules, we find $x = 15$, and a new matrix is obtained as given in table 4.10.

Table 4.10

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

Step 6. Now re-apply the test of step 3 to obtain the desired solution. Therefore, proceeding exactly in the same manner as in step 3 obtain the final table 4.11.

Table 4.11

15	⊗	20	15	0
15	15	0	10	⊗
15	0	20	15	5
0	15	20	⊗	5
5	⊗	10	⊗	⊗

It is observed that there are no remaining zeros, and every row (column) has an assignment. Since no two assignments are in the same column (they cannot be, if the procedure has been correctly followed), the 'zero assignment' is the required solution.

From original matrix (table 4.6), the minimum distance assignment is given by

Route	*A-e*	*B-c*	*C-b*	*D-a*	*E-d*	Total distance travelled
Distance (kms.)	200	130	110	50	80	570 kms.

Note : Table 4.11 may be obtained very quickly if we first apply step 2 and then step 1 in the original table 4.6.

Justification of rules used above in step 5 :

Justification of rules we have used in step 5 is based on the following two facts :

(i) The relative cost of assigning i-th facility to j-th job is not changed by the subtraction of a constant either from a column or from a row of the original effectiveness matrix.

(ii) An optimal assignment exists if the total reduced cost of the assignment is zero. This is the case when the minimum number of lines necessary to cover all zeros is equal to the order of the matrix. If however, it is less than n further reduction of the effectiveness matrix has to be undertaken.

The underlying logic can be explained with the help of table 4.9 in which only $3(= n - 2)$ lines can be drawn. Here an optimal assignment is not possible. So further reduction is necessary.

Further reduction is made by subtracting the smallest non-zero element 15 from all elements of the matrix table 4.9. This gives the following matrix :

Table 4.12

		L_1			
	15	–15	20	15	0
L_2	0	15	15	5	15
L_3	15	–15	20	15	5
	15	15	5	15	10
	6	–15	10	0	0

This matrix contains negative values. Since the objective is to obtain an assignment with reduced cost of zero, the negative numbers must be eliminated. This can be done by adding 15 to only those rows and columns which are covered by three lines (L_1, L_2, L_3) as shown above. In doing so the following change is noted.

Table 4.13

L_1

	15	(– 15 + 15)	20	15	0
L_2	(0 + 15)	[(– 15 + 15) + 15]	(– 15 + 15)	(– 5 + 15)	(– 15 + 15)
L_3	15 (– 15 + 15)	(– 15 + 15) [(– 15 + 15) + 15]	20 (– 15 + 15)	15 (– 15 + 15)	5 (– 10 + 15)
	5	(– 15 + 15)	10	0	0

This table is exactly the same as table 4.9. In fact, all this is the result of making the least non-zero element at the intersections, and subtracting from all uncovered elements, and leaving the other elements unchanged.

4.3.1 Assignment algorithm (Hungarian Assignment Method)

Various steps of the computational procedure for obtaining an optimal assignment may be summarized as follows :

Step 1. Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows.

Step 2. Further, modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus obtain the first modified matrix.

Step 3. Then, draw the minimum number of horizontal and vertical lines to cover all the zeros in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities :

(i) If $N = n$, the number of rows (columns) of given matrix, then an optimal assignment can be made. So make the zero assignment to get the required solution.

(ii) If $N < n$, then proceed to step 4.

Step 4. Determine the smallest element in the matrix, not covered by the N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the second modified matrix is obtained.

Step 5. Again repeat steps 3 and 4 until minimum number of lines become equal to the number of rows (columns) of the given matrix i.e., $N = n$.

Step 6. To make zero-assignment). Examine the rows successively until a row-wise exactly single zero is found, mark this zero by '*' to make the assignment. Then, mark a cross (×) over all zeros if lying in the column of the marked '*', showing that they cannot be considered for future assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for columns also.

If no unmarked zero is left, then the process ends. But if there be one or more unmarked zeros in any column or row, then mark '*' against one of the unmarked zeros arbitrarily and mark a cross in the cells of remaining zeros in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Step 8. Thus exactly one zero marked '*' is obtained in each row and each column of the matrix. The assignment corresponding to these '*' marked zeros will give the optimal assignment.

4.3.2 A rule to draw minimum number of lines

A very convenient rule of drawing minimum number of lines to cover all the 0's of the reduced matrix is given in the following steps :

Step 1. Tick (√) rows that do not have any marked (*) zero.

Step 2. Tick (√) columns having marked (*) zeros or otherwise in ticked rows.

Step 3. Tick (√) columns having marked 0's in ticked columns

Step 4. Repeat steps 2 and 3 until the chain of ticking is complete.

Step 5. Draw lines though all unticked rows and ticked columns.

This will give us the minimal system of lines.

4.4 Unbalanced Assignment Problem

If the cost matrix of an assignment problem is not a square matrix (number of sources is not equal to the number of destinations), the assignment problem is called as Unbalanced Assignment Problem. In such cases fictitious rows and/or columns with zero costs are added in the matrix so as to form a square matrix. Then the usual assignment algorithm can be applied to this resulting balanced problem.

Example 3. A company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated as follows (in hundreds of rupees):

Table 4.14

Jobs

		1	2	3	4	5
	1	2.5	5.0	1.0	6	1.0
	2	2.0	5.0	1.5	7	3.0
Machines	3	3.0	6.5	2.0	8	3.0
	4	3.5	7.0	2.0	9	4.5
	5	4.0	7.0	3.0	9	6.0
	6.	6.0	9.0	5.0	10	6.0

Solve the problem assuming that the objective is to minimize the total cost.

Solution : Introduce one more column for a fictitious job (say, job 6) in the cost matrix in order to get the following balanced assignment problem. The cost corresponding to sixth column are taken as zero.

Table 4.14

Jobs

		1	2	3	4	5	6
	1	2.5	5.0	1.0	6	1.0	0
	2	2.0	5.0	1.5	7	3.0	0
Machines	3	3.0	6.5	2.0	8	3.0	0
	4	3.5	7.0	2.0	9	4.5	0
	5	4.0	7.0	3.0	9	6.0	0
	6	6.0	9.0	5.0	10	6.0	0

The problem can now be solved by the discussed method.

4.5 Variation of the Assignment Problem

In this section, we shall discuss a variation of the assignment problem.

4.5.1 The Maximal Assignment Problem

Sometimes the assignment problem deals with the maximization of an objective function rather than minimizing it. For example, it may be required to assign persons to jobs in such a way that the expected profit is maximum. Such a problem may be solved easily by first converting it to a minimization problem and then applying the usual procedure of assignment algorithm. This conversion can be very easily done by subtracting from the highest element all the elements of the given profit matrix; or equivalently, by placing minussign before each element of the profit-matrix in order to make it a cost-matrix.

Following examples will make the procedure clear.

Example 4. (Maximization problem). A company has 5 jobs to be done. The following matrix shows the return in rupees on assigning i-th ($i = 1, 2, 3, 4, 5$) machine to the j-th job ($j = A, B, C, D, E$). Assign the five jobs to the five machines so as to maximize the total expected profit.

Table 4.15

Jobs

		A	B	C	D	E
	1	5	11	10	12	4
	2	2	4	9	3	5
Machines	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Solution : Step 1. (Converting from maximization to minimization problem) :

Since the highest element in the matrix is 14, so subtracting all the elements from 14, the following reduced cost (opportunity loss of maximum profit) matrix is obtained.

Table 4.16

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	2

Step 2. Now following the usual procedure of solving an assignment problem, an optimal assignment is obtained in the following table :

Table 4.17

1	⊗	0*	⊗	5
⊗	13	⊗	5	0*
5	1	7	0*	5
3	0*	9	4	5
0*	3	3	1	5

This table gives the optimum assignment as : $1 \to C, 2 \to E, 3 \to D, 4 \to B, 5 \to A$ with maximum profit of Rs. 50.

4.6 Questions

1. Discuss the similarity between a transportation problem and an assignment problem.

2. Discuss the steps of Hungarian method for solving an assignment problem.

3. Five different jobs are to be assigned to five operators. The following matrix gives the processing times in hours;

		Operator				
		1	2	3	4	5
	1	5	6	8	6	4
	2	4	8	7	7	5
Job	3	7	7	4	5	4
	4	6	5	6	7	5
	5	4	7	8	6	8

Find the optimal job assignment.

4. How should four sales persons be assigned to four sales regions when the annual sale figures (in crores of rupees) for different salesmen working in different sales regions are as below :

		Sales region			
		1	2	3	4
	1	5	11	8	9
	2	5	7	9	7
Salesman	3	7	8	9	9
	4	6	8	11	12

5. The flight timings between two cities *X* and *Y* are given in the following two tables. The minimum layover time of may crew in either of the cities is 3 hours. Determine the base city for each crew so that the sum of the layover times of all the crews in non-base cities is minimized.

Timings of Flights from City *X* to City *Y*

Flight number	Departure time (from City *X*)	Arrival time (to City *Y*)
101	6 a.m.	8 a.m.
102	10 a.m.	12 noon
103	3 p.m.	5 p.m.
104	8 p.m.	10 p.m.

Timings of Flights from City *Y* to City *X*

Flight number	Departure time (from City *Y*)	Arrival time (to City *X*)
201	5.30 a.m.	7 a.m.
202	9 a.m.	10.30 a.m.
203	4 p.m.	5.30 p.m.
204	10 p.m.	11.30 p.m.

Unit 5 ❑ Theory of Games

Structure

5.1 Introduction

Many practical problems require decision-making in a competitive situation, where there are two or more opponents with conflicting interests and the action of one opponent depends upon the ones taken by the others. For example, candidates for an election, advertising and marketing campaigns by competing business firms, countries involved in military battles, etc. have their conflicting interests. In any competitive situation the courses of action available to a competitor may be finite or infinite. A competitive situation with a finite number of competitors each having a finite number of courses of action is termed as a "competitive game".

A competitives game has the following properties—

(a) There are a finite number of competitors, also known as players.

(b) Each of the competitors has available to him a finite list of possible courses of action or strategies; this list may not be same for all players.

(c) A play of the game results when each of the players chooses a course of action from his list. The choices are assumed to be made simultaneously so that, no player knows his opponents' choices until he is already committed.

(d) The outcome of a play is affected by the particular set of courses of action adopted by the players. Each outcome determines a set of payments (positive, negative or zero), one to each player.

5.2 Two-person zero-sum games

Games having the "zero-sum" character, i.e., the algebraic sum of the gains and losses of all the players add up to zero, are called zero-sum games. Otherwise, they are referred to as nonzero-sum games. Zero-sum games with two players are called two-person zero-sum games. In such a game the gain (loss) of one player is exactly equal to the loss (gain) of the other player.

The assumptions gaverning a two-person zero-sum game will, therefore, be as follows—

(a) There are exactly two players, with opposite interests.

(b) The number of courses of action available to each player is finite. The list may not be same for the two players.

(c) For each course of action selected by a player there results a payoff.

(d) For each play of the game, the amount won by a player is exactly equal to the amount lost by the other.

Let m and n denote the number courses of action available to the two players, say A and B respectively. The respective payoffs may be summarized in a $m \times n$ matrix usually referred to as the payoff matrix of the game. This payoff matrix gives the payoffs to one of players, say A, corresponding to the different courses of action adopted by the two players. A positive payoff will mean gain to A and loss to B, while a negative payoff will mean loss to A and gain to B. A typical payoff matrix in terms of payoffs to A will be as follows :

Table 5.1

Player B

	B_1	B_2	......	B_n
A_1	c_{11}	c_{12}	......	c_{1n}
A_2	c_{21}	c_{22}	......	c_{2n}
Player A	...			

A_m	c_{m1}	c_{m2}	......	c_{mn}

Here $\{A_1, A_2, ..., A_m\}$ are the courses of action of player A, $\{B_1, B_2, ..., B_n\}$ are the courses of action of play B, and c_{ij} = payoff to A when A adopts the course of action A_i and B adopts B_j, $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

Then, $-c_{ij}$ will be the payoff to B when A adopts A_i and B adopts B_j, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

Example 1. Consider a two-person coin tossing game. Each player tosses an unbiased coin simultaneously. Player B pays Rs. 7 to player A if $\{H, H\}$ occurs and Rs. 4 if $\{T, T\}$ occurs. Otherwise, A pays Rs. 3 to B. Here $H \equiv$ head, $T \equiv$ tail. This will be a zero-sum game since the winning of one player is the loss of the other. The payoff matrix in terms of payoffs to player A will be

		Player B	
		H	T
Player A	H	7	− 3
	T	− 3	4

The payoff matrix in terms of payoffs to player B will, therefore, be

		H	T
Player B	H	− 7	3
	T	3	− 4

Strategy : A competitive game is sometimes referred to as a game of strategy. When we talk of the strategy adopted by a player in a competitive game we mean a set of rules which tells the player course of action he should take in a play of the game. This strategy may be of two kinds—

(a) **Pure strategy :** A pure strategy is a decision rule which tells the player, in advance of all plays, to choose a particular course of action for each play. A pure strategy is identified by the name representing the course of action chosen.

(b) **Mixed strategy :** A mixed strategy is a decision rule which tells the player, in advance of all plays, to choose a course of action for each play in accordance with some probability distribution. For example, if a player has two courses of action but chooses a particular one for each play, then he has a pure strategy. But for each play if he flips an unbiased coin to decide upon which course of action he should take, he has a mixed strategy.

By solving a game we mean finding the "optimal strategies" for both players and the corresponding "value of the game".

5.3 Minimax criterion and optimal strategy

The "minimax criterion of optimality" states that if a player lists the worst possible outcomes of all his potential strategies, he will choose that strategy to be most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an "doptimal strategy".

Example 2. Consider the following two-person, zero-sum game matrix which represents payoff to the player *A*. Find the optimal strategy, if any.

		B		
		I	II	III
	I	-3	-2	6
A	II	2	0	2
	III	5	-2	-4

Solution : The player *A* wishes to obtain the largest possible 'c_{ij}' by choosing a course of action from his list (I, II, III) while the player *B* is determined to make *A*'s gain the minimum possible by choice of a course of action from his list (I, II, III). The player *A* is called the maximizing player and *B* the minimizing player.

If the player *A* chooses the 1st course of action, then it could happen that the player *B* also chooses his 1st course of action. In this case the player *B* can guarantee a gain of at least -3 to player *A*, i.e.

$$\min\{-3, -2, 6\} = -3$$

Similarly, for other choices of player *A*, i.e., II and III, *B* can force the player *A* to get only 0 and -4, respectively by his proper choices from (I, II, III), i.e.,

$$\min\{2, 0, 2\} = 0 \text{ and } \min\{5, -2, -4\} = -4.$$

The minimum value in each row guaranteed to the player A is indicated by 'row minimum' in table below. The best choice for the player A is one which maximizes his least gains $-3, 0, -4$, and as max $\{-3, 0, -4\} = 0$, the optimum course of action which assure at most the gain 0 is II.

Table 5.2

			B		
		I	II	III	Row minimum
	I	− 3	− 2	6	− 3
A	II	2	0	2	0 Maximin
	III	5	− 2	− 4	value (v) = 0
Column maximum		5	0	6	− 4

Minimax Value $(\bar{v}) = 0$

In general, the player A should try to maximize his least gains or to find out $\max_i \min_j c_{ij}$.

$$\max_i\left[\min_j\{c_{ij}\}\right] = \min_j\left[\max_i\{c_{ij}\}\right] = c_{rs}\,(say)$$

Player B, on the other hand, can argue similarly to keep A's gain the minimum. He realizes that if he plays his 1st pure strategy, he can loose no more than $5 = \max\{-3, 2, 5\}$ regardiess of A's selection. Similar arguments can be applied for remaining pure strategies II and III. Corresponding results are indicated in the table by 'column maximum'. The player B will then select the pure strategy that minimizes his maximum loses. This is given by the II and his corresponding loss is given by

$$\min\{5, 0, 6\} = 0$$

The player A's selection is called the maximin strategy and his corresponding gain is called the maximin value or lower value $(\bar{v})$ of the game. The player B's selection is called the minimax strategy and his corresponding loss is called the minimax value or upper value (v) of the game. The selections made by player A and B are based on the so called minimax (or maximin) criterion. It is seen from the governing conditions that the minimax (upper) value $\bar{v}$ is greater than or equal to the maximin (lower) value $\underline{v}$ (see Theesem). In the case where equality holds i.e.,

$$\max_i \min_j c_{ij} = \max_j \min_i c_{ij} \quad \text{or,} \quad \underline{v} = \bar{v}, \qquad \text{... (1)}$$

the corresponding pure strategies are called the optimal strategies and the game is said to have a saddle point. It may not always happen as shown in the following example.

Note : For convenience, the minimum values are shown by '()' and maximum values by [] in the table.

Theorem 1. Let (c_{ij}) be the payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\bar{v}$ the minimax value of the game $\bar{v} \geq \underline{v}$. That is $\min_i \left[\max_i \{c_{ij}\} \right] \geq \max_j \left[\min_j \{c_{ij}\} \right]$.

Proof. We have $\max_i \{c_{ij}\} \geq c_{ij}$ for any j, and $\min_j \{c_{ij}\} \leq c_{ij}$ for any i.

Let the above maximum be attained at $i = i^*$ and the minimum be attained at $j = j^*$. So $c_{i^*j} > c_{ij} > c_{ij^*}$ for any i and j.

This implies that

$$\min_j \{c_{i^*j}\} \geq c_{ij} \geq \max_i \{c_{ij^*}\} \text{ for any } i \text{ and } j.$$

Hence $$\min_j \left[\max_i \{c_{ij}\} \right] \geq \max_i \left[\min_j \{c_{ij}\} \right], \text{ or } \bar{v} \geq \underline{v}.$$

5.4 Saddle point, optimal strategies and value of game

Some Definitions.

Saddle Point : A saddle point of a payoff matrix is the position of such an element in the payoff matrix which is minimum in its row and maximum in its column.

Mathematically, if a payoff matrix $\{c_{ij}\}$ is such that

$$\max_i \left[\min_j \{c_{ij}\} \right] = \min_j \left[\max_i \{c_{ij}\} \right] = c_{rs} (say),$$

then the matrix is said to have a saddle point (r, s).

Optimal strategies. If the payoff matrix $\{c_{ij}\}$ has the saddle point (r, s), then the players (A and B) are said to have their r-th and s-th strategies, respectively, as their optimal strategies.

Value of game. The payoff (c_{rs}) at the saddle point (r, s) is called the value of game and it is obviously equal to the maximin ($\underline{v}$) and minimax value ($\bar{v}$) of the game.

A game is said to be a fair game if $\bar{v} = \underline{v} = 0$. A game is said to be strictly determinable if $\bar{v} = \underline{v}$.

Note : A saddle point of a payoff matrix is, sometime, called the equilibrium point of the payoff matrix.

In Example 1, $\bar{v} = \underline{v} = 0$. This implies that the game has a saddle point given by the position (2, 2) in the payoff matrix. The value of the game is thus equal to zero and both players select their strategy as the optimal strategy. In this example, it is also seen that no player can improve his position by choosing any other strategy.

Rules for determining a saddle point :

1. Select the minimum element of each row of the payoff matrix and mark them by '()'.
2. Select the greatest element of each column of the payoff matrix and mark them by '[]'.
3. If there appears an element in the payoff matrix marked by '()' and '[]' both, the position of that element is a saddle point of the payoff matrix.

Solution of games with saddle points :

To obtain a solution of a two-person game, often referred to as a rectangular game, one has to find out :

(i) the best strategy for player A;

(ii) the best strategy for player B, and

(iii) the value of the game (c_{rs}).

It is already seen that the best strategies for players A and B will be those which correspond to the row and column, respectively, through the saddle point. The value of the game to the player A is the element at the saddle point, and the value to the player B will be its negative.

Example 3. Player A can choose his strategies from $\{A_1, A_2, A_3\}$ only, while B can choose from the set (B_1, B_2) only. The rules of the game state that the payments should be made in accordance with the selection of strategies :

Strategy pair selected	Payments to be made	Strategy pair selected	Payments to be made
(A_1, B_1)	Payer A pays Re. 1 to player B	(A_2, B_2)	Player B pays Rs. 4 to player A
(A_1, B_2)	Player B pays Rs. 6 to player A	(A_3, B_1)	Player A pays Re. 1 to player B
(A_2, B_2)	Player B pays Rs. 2 to player A	(A_3, B_2)	Player A pays Rs. 6 to player B

What strategies should A and B play in order in order to get the optimum benefit of the play?

Solution : With the help of above rules the following payoff matrix is constructed:

		Player B	
		B_1	B_2
	A_1	-1	6
A	A_2	2	4
	A_3	1	16

In Table 5.3 the payoffs marked '()' represent the minimum payoff in each row and those marked '[]' represent the maximum payoff in each column of the payoff matrix.

Obviously, the matrix has a saddle point at position (2, 1) and the value of the game is 2.

Table 5.3

	Player B	
	B_1	B_2
A_1	(– 1)	[6]
A_2	[(2)]	4
A_3	(1)	[6]

Thus, the optimum solution to the games is given by :

(i) the optimum strategy for player A is A_2;

(ii) the optimum player A strategy for player B is B_1 : and

(iii) the value of the game v is Rs. 2 for player A and Rs. (– 2) for player B.

Also, since $v = 0$, the game is not fair, although it is strictly determinable.

Example 4. The payoff matrix of a game is given. Find the solution of the game to the player A and B.

		B				
		I	II	III	IV	V
	I	– 2	0	0	5	3
A	II	3	2	1	2	2
	III	– 4	– 3	0	– 2	6
	IV	5	3	– 4	2	– 6

Solution : First find the saddle point by putting first bracket around each row minimum and putting square bracket around each column maximum.

The saddle point thus obtained is shown in Table 5.4.

Table 5.4

Optimum strategy for B

		I	II	III	IV	IV		
	I	(– 2)	0	0	[5]	3	– 2	
Optimum strategy	II	3	2	[(1)]	2	2	1	Maximin value (v) = 1
for *A*	II	(– 4)	– 3	0	– 2	[6]	– 4	
	III	[5]	[3]	(– 4)	2	(– 6)	– 6	
	IV	5	3	1	5	6		

Column maximum

Minimax value ($\underline{v}$) = 1

Hence, the solution to this game is given by

(i) the best strategy for player A is II;

(ii) the best strategy for player B is III; and

(iii) the value of the game is 1 to player A and – 1 to player B.

Example 5. Solve the game whose payoff matrix is given by

$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} -2 & 15 & -2 \\ -5 & -6 & -4 \\ -5 & 20 & -8 \end{bmatrix} \end{array}$$

Solution : Table 5.5 may be formed as explained earlier.

This game has two saddle points in positions (1, 1) and (1, 3). Thus, the solution to this game is given by.

(i) the best strategy for the player A is I;

(ii) the best strategy for the player B is either I or III, i.e., the player B can use either of the two strategies (I, III); and

(iii) the value of the game is 2 for player A and – 2 for player B.

Table 5.5

		I	II	III	Row minimum	
	I	[(– 2)]	15	[(– 2)]		Maximin value ($\underline{v}$) = – 2
A	II	– 5	(– 6)	– 4	– 2	
	III	– 5	[20]	(– 8)	– 6	
Column max		– 2	20	– 2	– 8	

Minimax value ($\overline{v}$) = – 2.

5.5 Questions

1. What is a game? Explain the following terminologies in relation to game theory : (a) players, (b) strategy, (c) minimax principle, (d) saddle point, (e) value of a game, (f) two-person zero-sum game.

2. Two machines-tool companies *A* and *B*, competing for supplying a *CNC* lathe to a new factory. Each company has listed its alternatives/strategies for selling machine tools.. The strategies of company *A* are listed below :

(a) giving special price
(b) giving 15% worth of additional tools
(c) supplying some work holding device free of cost

The strategies of company *B* are as follows :

(a) giving special price
(b) giving 20% worth of additional tools
(c) giving free training to the users of the organization which is buying the machine. The estimated gains (+)/losses (–) in lacs of rupees of company *A* for various possible combinations of the alternatives of both the companies are summarized in the following table. Find the optimal strategies for the companies.

		Company *B*		
		1	2	3
Company *A*	1	40	45	50
	2	20	45	60
	3	25	30	30

3. Find the optimal strategies of the players in the following game :

		Player *B*		
		1	2	3
Player *A*	1	30	20	40
	2	55	50	60
	3	60	30	40

Unit 6 ❑ Project Management : PERT and CPM

Structure

6.1 Introduction

In Operations Research networks play an important role as quite often the problem of determining an optimal solution can be looked upon as the problem of selecting the best sequence of operations out of a finite number of available alternatives that can be represented as a network. Network scheduling is a technique used for planning and scheduling large projects in the fields of construction, maintenance, fabrication, purchasing, computer system installation, research and development designs etc. The technique involves minimizing trouble spots, such as production bottlenecks, delays and interruptions, by determining critical factors and coordinating various parts of a job.

There are two basic planning and controlling techniques that use a network to complete a predetermined project or schedule. They are the Program Evaluation and Review Technique (PERT) and the Critical Path Method (CPM). Several variations of these have also been developed, one such variation being the Review Analysis of Multiple Projects (RAMP) which is useful in guiding the "activities" of several projects at one time.

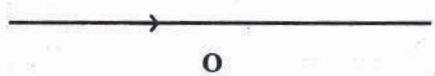
6.2 Some basic concepts

1. **Activity :** An activity is a task, or part of work to be done, that consumes time, effort, money or other resources. In a diagrammatic representation, an activitiy is represented by an arrow in the direction of progress of the project.

(a) **Critical activity :** An activity is said to be critical if a delay in its start causes a delay in the completion time of the entire project.

(b) **Non-critical activity :** A non-critical activity is such that the time the time between its earliest start and its latest completion dates (as allowed by the project) (defined later) is longer than its actual duration. In this case the non-critical activity is said to have a slack or float time.

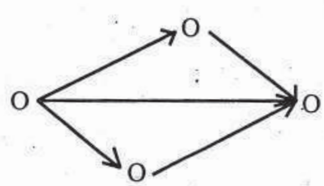
2. **Node :** The specific point of time at which an activity begins or ends is called a node. The starting time point is called the tail node and the endidng point is the head node. A node is generally represented by a circle.



Node

Nodes may be numbered in a network.

3. **Network :** A network is a graphical representation of a project's operations and is composed of activities and nodes.



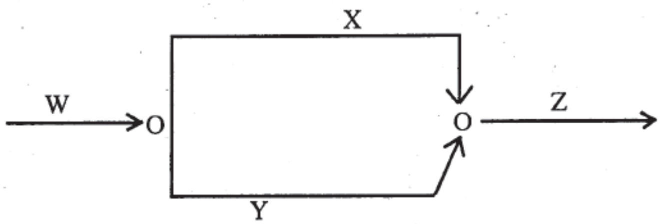
Network

In a network, no two activities should be identified by the same head and tail nodes.

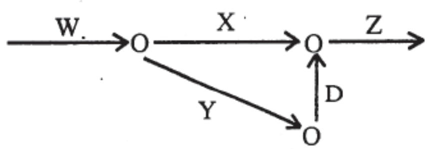
Critical path : The dcritical path is the shortest path through a network. It gives a chain of critical activities that connect the start and end nodes of the network. The critical path is generally denoted by double lines in a network.

Precedence relationship in network : A project may be considered as a series of activities in which an activity may begin only after another activity or activities have finished. In a network schedule this type of relationship is called precedence relationships and is represented by inequality. For example, $X < Y$ denotes that the activity X must be completed before the start of activity Y.

Two or more activities may be performed concurrently. For example in the figure below activities X and Y have the same nodes.



In such a situation a dummy activity is introduced either between X and one of the nodes, or between Y and one of the nodes. This results in the following representation—



Here D is the dummy activity introduced. As a result, activities X and Y can be identified by unique head nodes. It should be noted that a dummy activity does not consume any time or resource.

Note : Dummy activities may also used to establish correct precedence relationships. For example, suppose in a certain project activities A and B precede C while E is preceded by B only. Figure 1 represents the incorrect presentation since the diagram implies that E is preceded by both A and B. Figure 2 shows use of dummy activity D for the correct representation.

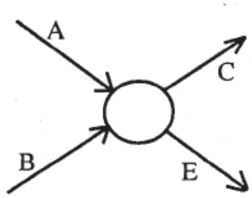


Fig. : 1

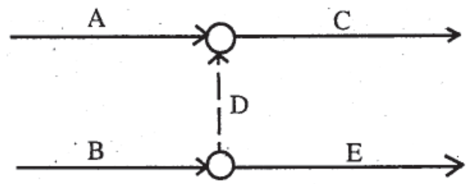


Fig. : 2

To ensure the correct precedence relationship in a network, one should be aware of the following for each activity—

(a) which activities must be completed immediately before the activity;

(b) which activities must follow the activity;

(c) which activities must occur concurrently with the activity.

Example 1. Construct the network comprising the activities *A, B, C* .... and *L* such that the following precedence relationships are satisfied—

(i) *A, B* and *C*, the first activities can start simultaneously.

(ii) $A, B < D$.

(iii) $B < E, F, H$.

(iv) $F, C < G$.

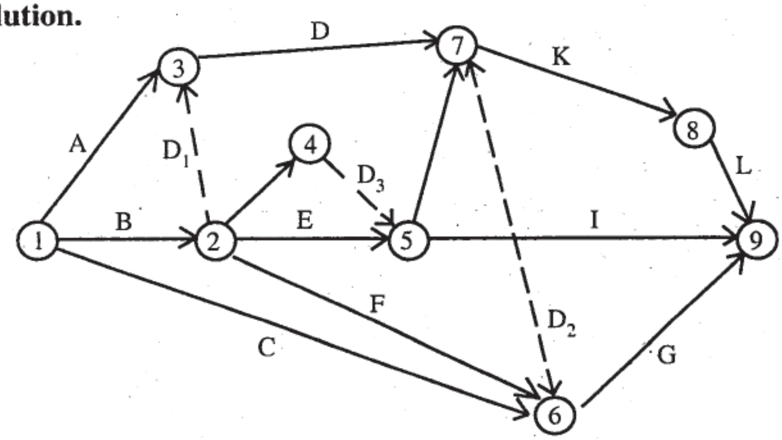
(v) $E, H < I, J$.

(vi) $C, D, F, J < K$.

(vii) $K < L$.

(viii) *I, G* and *L* are the terminal activities of the project.

Solution.



Dummy activities D_1 and D_2 are used to establish correct precedence relationship. and D_3 is used to identify activities E and H with unique end nodes. The nodes are numbered such that their ascending order indicates the direction of progress in the project.

6.3 Calculation of the critical path

The critical path calculation consists of two phases called forward pass and backward pass.

Forward pass : Here for each activity we calculate the earliest starting time (ES) of the activity, i.e., the earliest possible time when the activity can begin, assuming that all its predecessors also start at their ES.

Let the project start at time $t = 0$. Let ES_i be the earliest start time of all activities emanating from node i. If $i = 0$ be the starting node then conventionally we take $ES_0 = 0$. Let t_{ij} be the duration of an activity (i, j). (Activity (i, j) means an activity with tail node i and head node j). Then, for node j,

$$ES_j = \{ES_i + t_{ij}\}, \text{ for all } (i, j) \text{ activites defined, where } ES_0 = 0.$$

This also defines the earliest finishing time of an activity ending at the node j.

ES_i's are represented in the network enclosed in squares ([]) alongside the nodes.

Backward pass : Here for each activity we calculate the latest finishing time, or latest completion time (LC). This also defines the latest start-time of an activity starting from a node. Let LC_i be the LC for all activities with head node i. If $i = n$, the terminal node, then $LC_n = ES_n$. For each node i,

$$LC_i = \min_j \{LC_j - t_{ij}\}$$

for all (i, j) activities defined. LC_i's are usually represented in the network enclosed in triangles Δ alongside the nodes.

The critical activities can now be identified. An activity (i, j) will lie on the critical path if it satisfies the following three conditions :

(i) $ES_i = LC_i$.

(ii) $ES_j = LC_j$.

(iii) $ES_j - ES_i = LC_j - LC_j - LC_i = t_{ij}$.

All these conditions show that there is no float or slack time between earliest start (finish) time and the latest start (finish) time of the activity. In the network, these activities are characterized by the numbers in ❑ and Δ being the same at each of the tail and head nodes and that the difference between the number in ❑ (or Δ) at the head node and the number in ❑ (or Δ) at the tail node is equal to the duration of the activity.

6.4 Determination of the Floats or Slacks

For any activity (i, j) the earliest start time is ES_i, the earliest finish time is ES_j, the latest start time is LC_i and the latest finish time is LC_j. Then the total float or slack available for the activity is given by

$$TF_{ij} = LC_i - ES_i = LC_j - ES_j.$$

6.5 Critical Path Method (CMP)

The critical path method for finding the critical path in a given project may be summarized in the following steps :

Step 1. List all the activities (jobs) and draw the network.

Step 2. Indicate the activity times in parenthesis above the arrow representing the activity besides the name (generally given by an alphabet) of the activity.

Step 3. Calculate earliest start time and earliest finish time for each node and put it in ❑ against its tail node. Also calculate latest finish time for each activity and put it in Δ against its head node.

Step 4. Prepare a table which includes an activity (i, j)'s normal time, earliest times and latest times. Determine the slack or float for each activity.

Step 5. Note down the critical activities with zero slack and connect them with starting node and ending node in the network using double lines. This gives the critical path.

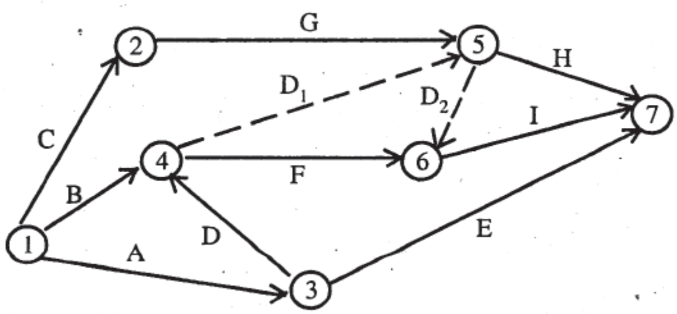
Example 2. A project consists of a series of tasks labeled *A, B, ..., H, I*. Using the notations $W < X, Y$ to mean that X and Y can not start until W is completed, and $X, Y < W$ to mean W cannot start until both X and Y are completed, construct the network diagram having the following constraints :

$$A < D, E;\ D < F;\ C < G;\ G < H;\ F;\ G < I.$$

Find also the minimum time of completion of the project, when the time in days of completion of each tasks is as follows :

Task	:	A	B	C	D	E	F	G	H	I
Time	:	23	8	20	16	24	18	19	4	10

Solution : Using the given constraints, the resulting network is shown in the figure below. The dummy activities D_1 and D_2 are introduced to establish the correct precedence relationships. The nodes of the project are numbered in such a way that their ascending order indicates the direction of progress in the project.



To determine the minimum time of completion of the project (critical path), we compute ES_j and LS_i for each of the task (i, j) of the project. The critical path calculations as applied to figure are as follows :

$$ES_1 = 0$$

$$ES_2 = ES_1 + t_{13} = 0 + 20 = 20$$

$$ES_3 = ES_1 + t_{14} = 0 + 23 = 23$$

To obtain the value of ES_4, since there are two incoming tasks (1, 4) and (3, 4), we shall have

$$ES_4 = \max_{i=1,3} \{ES_i + t_{i4}\}$$

$= \max.\{0 + 8, 23 + 16\} = 39.$

The procedure is repeated until ES_j is computed for all j. Thus

$$ES_5 = \max_{i=2,4} \{ES_i + t_{i5}\} = \max.\{39 + 0, 20 + 19\} = 39$$

$$ES_6 = \max_{i=4,5} \{ES_i + t_{i6}\} = \max.\ \{39 + 18, 39 + 0\} = 57$$

$$ES_1 = \max_{i=3,5,6} \{ES_i + t_{ii}\} = \max.\ \{23 + 24, 39 + 4, 57 + 10\} = 67$$

The values of LC_i are now obtained, these are

$$LC_7 = EC_7 = 67$$

$$LC_6 = LC_6 - t_{67} = 67 - 10 = 57$$

$$LC_5 = \min_{j=6,7} \{LC_j - t_{5j}\} = \min\ \{57 - 0, 67 - 4\} = 57$$

$$LC_4 = \min_{j=5,6} \{LC_j - t_{4j}\} = \min\{57 - 0, 57 - 18\} = 39$$

$$LC_3 = \min_{j=4,7} \{LC_j - t_{3j}\} = \min\{39 - 16, 67 - 24\} = 23$$

$$LC_2 = \{LC_5 - t_{25}\} = 57 - 19 = 38$$

$$LC_1 = \min_{j=2,3,4} \{LC_j - t_{1j}\} = \min\{38 - 20, 23 - 23, 39 - 8\} = 0.$$

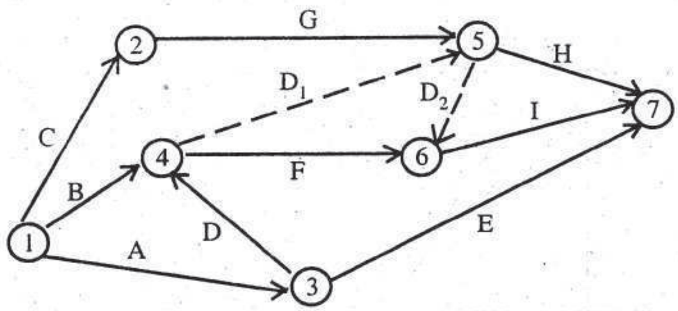
To evaluate the critical nodes, all these calculations are put in the following table:

Table 6.1

Task (i, j)	Normal time (t_{ij})	Earlest time		Latest time		Stock available TF_{ij}
		Start ES_i	Finish ES_j	Start LC_i	Finish LC_j	
(1, 2)	20	0	20	18	38	18
(1, 3)	23	0	23	0	23	0
(1, 4)	8	0	8	31	39	31
(2, 5)	19	20	39	38	57	18
(3, 4)	16	23	39	23	39	0
(3, 7)	24	23	47	43	67	20

Task (i, j)	Normal time (t_{ij})	Earlest time		Latest time		Stock available TF_{ij}
		Start ES_i	Finish ES_j	Start LC_i	Finish LC_j	
(4, 5)	0	39	39	57	57	18
(4, 6)	18	39	57	39	57	0
(5, 6)	0	39	39	57	57	18
(5, 7)	4	39	43	63	67	24
(6, 7)	10	57	67	57	67	0

The above table shows that the critical nodes are for the tasks (1, 3), (3, 4), (4, 6) and (6, 7).



Critical path (denoted by the bold lines)

It is apparent from the above that the critical path comprises the activities (1, 3), (3, 4), (4, 6) and (6, 7). This path represents the shortest possible time to complete the entire project.

6.6 PERT

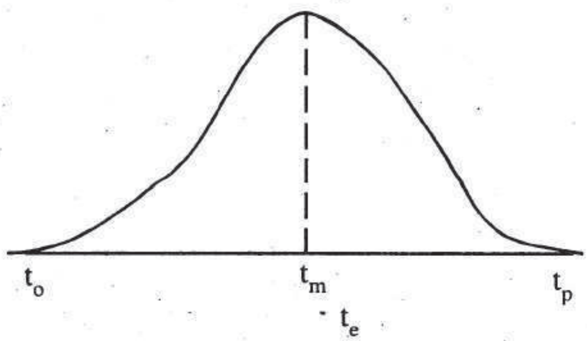
The network method discussed so far may be termed as deterministic in the sense that the estimated activity times (i.e., times of completion of activities) are assumed to be non-stochastic. A more realistic situation would be where activity times are random variables following some probabilistic laws. Probability considerations are incorporated by assuming that the time estimate of each activity is based on the following three values :

t_o = the optimistic time, which is the shortest time to complete the activity if execution goes very well;

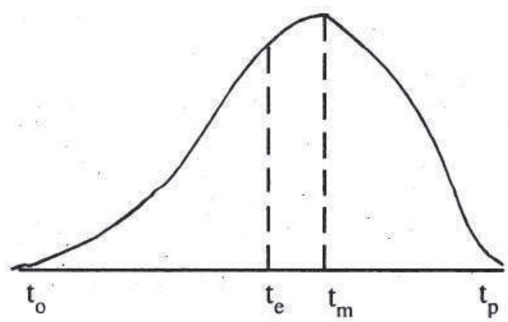
t_p = the pessimistic time, which is the longest time to complete the activity, if everything goes wrong;

t_m = the most likely time, which is required if the execution is normal.

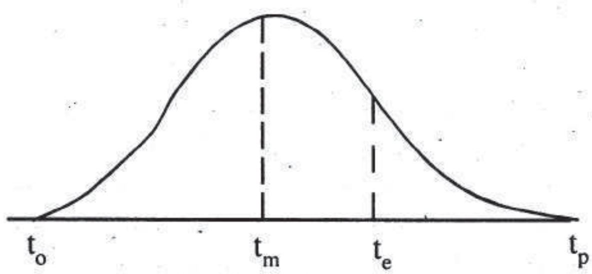
The range specified by the estimated optimistic and pessimistic times to and t_p, respectively supposedly must enclose every possible estimate of the duration of the activity. The most likely estimate t_m need not coincide with the midpoint $(t_o + t_p)/2$ and may occur to its left or its right. Because of these properties, it is intuitively justified that the duration for each activity may follow a beta distribution with its unimodal point occurring at t_m and its end points t_o and t_p. The figure below shows the three cases of the beta distribution, which are (a) symmetric, (b) skewed to the right, and (c) skewed to the left.



(a) symmetric



(b) skewed to right



(c) skewed to left

The expressions for the mean t_e and variance σ^2 of the beta distribution are developed in the following way. t_e is taken to be the average of the mid-point $(t_o + t_p)/2$ and $2t_m$, i.e.,

$$t_e = \frac{1}{3}[(t_0 + t_p)/2 + 2t_m] = \frac{1}{6}[(t_0 + t_p + 2t_m]$$

The range (t_o, t_p) is assumed to enclose about 6 standard deviations of the distribution, since 90% or more of any probability density function lies within 3 standard deviations of its mean. Thus,

$$\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$$

It is now possible to estimate the probability of occurrence of each node in the network. Let μ_i be the earliest time of node i. Since the times of activities summing up to i are random variables, μ_i is also a random variable. Assuming that all the activities in the network are statistically independent, we obtain the mean and variance of μ_i as follows. If there is only one path leading from the starting node to node i, $E(\mu_i)$ is given by the sum of the expected times t_c for the activities along this path, and var(μ_i) is the sum of the variances of the same activities. Complications arise when more than one path leads to the same node. In this case to compute exact $E(\mu_i)$ and var(μ_i) one must first develop the statistical distribution for the longest of the different paths. This task is rather difficult in general and simplifying assumption is introduced that computes $E(\mu_i)$ and var(μ_i) as

$$E(\mu_i) = ES_i,\ \text{var}(\mu_i) = \sum_k V_k,$$

where k defines the activities along the longest path leading to node i and V_k denotes the variance of the k-th activity on this path.

Since μ_i is the sum of independent random variables (viz. the durations of activities on path leading to node i), by Central Limit Theorem, μ_i is approximately normally distributed with mean $E(\mu_i)$ and variance var(μ_i). Since μ_i represents the earliest occurrence of node i, the node i will meet a certain scheduled time ST_i (specified by the analyst) with probability

$$P(\mu_i \le ST_i) = P\left(\frac{\mu_i - E(\mu_i)}{\sqrt{\text{var}(\mu_i)}} \le \frac{ST_i - E(\mu_i)}{\sqrt{\text{var}(\mu_i)}}\right)$$
$$= P(z \le D_i),$$

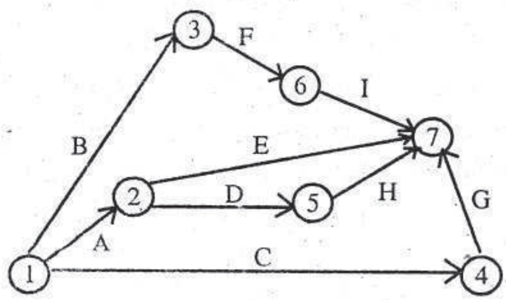
where z is the standard normal variate with mean 0 and variance 1, and

$$D_i = \frac{ST_i - E(\mu_i)}{\sqrt{\text{var}(\mu_i)}}$$

It is a common practice to calculate the above probability. Such probabilities represent the chance that the succeeding nodes will occur within (ES_i, LC_i) durations.

Example 3 : A project is represented by the network in the figure below and has the following data :

Task	A	B	C	D	E	F	G	H	I
Latest time	5	18	26	16	15	6	7	7	3
Greatest time	10	22	40	20	25	12	12	9	5
Most like time	8	20	33	18	20	9	10	8	4



Determine the following :

(a) expected activity times and their variance;

The earliest and latest expected times to reach each node;

(b) The critical path, and

(c) The probability of a node occurring at the proposed completion date if the original contract time of completing the project is 41.5 weeks.

Solution : (a) The expected activity time, t_e, is calculated by using the three given estimated times in the relation

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

The variance, σ^2, for the activities is calculated by the formula

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

The following table provides the required information regarding t_e and σ^2.

Table 6.2

Activity	t_o	t_p	t_m	t_e	σ^2
(1, 2)	5	10	8	7.8	0.694
(1, 3)	18	22	20	20.0	0.444
(1, 4)	26	40	33	33.0	5.429
(2, 5)	16	20	18	18.0	0.443
(2, 6)	15	25	20	20.0	2.780
(3, 6)	6	12	9	9.0	1.00
(4, 5)	7	12	10	9.8	0.694
(5, 7)	7	9	8	8.0	0.111
(6, 7)	3	5	4.	4.0	0.111

(b) T_e or $E\{\mu\}$, the earliest expected time for each node, is obtained by taking the sum of the expected times for all the activities leading to the node. When more than one activity leads to a node i, the greatest of $E\{\mu_i\}$ is chosen. Thus, we have

$$E\{\mu_1\} = 0,$$
$$E\{\mu_2\} = 0 + 7.8 = 7.8.$$
$$E\{\mu_3\} = 0 + 20.0 = 20.0,$$
$$E\{\mu_4\} = 0 + 33.0 = 33.0,$$
$$E\{\mu_5\} = 7.8 + 18.0 = 25.8,$$
$$E\{\mu_6\} = \text{max.}\{7.8 + 20.0, 20.0 + 9.0\} = 29.0,$$
$$E\{\mu_7\} = \text{max.}\{33.0 + 9.8, 25.8 + 8.0, 29.0 + 4.0\} = 42.8.$$

For the lateast expected times $T_L = E(L)$, we start with T_L equal to $T_e = E(\mu)$ for the last node. Now for each path move backwards, subtracting the "t_e" for each activity link. Thus, we have

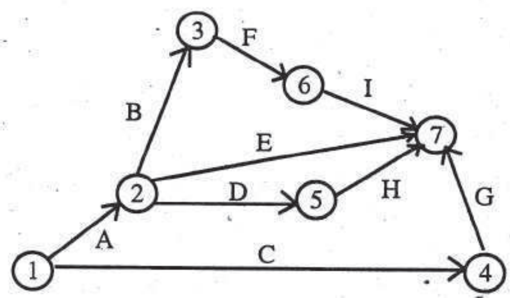
$$T_{L7} = E\{L_7\} = 42.8,$$
$$T_{L6} = E\{L_6\} = 42.8 - 38.8,$$
$$T_{L5} = E\{L_5\} = 42.8 - 8.0 = 34.8,$$
$$T_{L4} = E\{L_4\} = 42.8 - 9.8 = 33.0,$$

$$T_{L3} = E\{L_3\} = 38.8 - 9.0 = 29.8,$$
$$T_{L2} = E\{L_2\} = \min.\{34.8 - 18.0, 38.8 - 20.0\} = 16.8,$$
$$T_{L1} = E\{L_1\} = \min.\{16.8 - 7.8, 29.8 - 20.0, 33.0 - 33.0\} = 0.$$

(c) For the critical path, we calculate the slack time by taking the difference between the earliest expected times and latest allowable times, that is, $T_s = T_L - T_e$. Once the slack times are known to us, the critical path may determined by finding the path with zero slack. The critical path for the problem under consideration is indicated in the figure below by the dark double line. The following table gives the calculation for T_e, T_L and T_s.

Table 6.3

Node i	t_e	$E\{\mu_i\} = T_e$	$E\{L_i\} = T_L$	T_s	$\text{Var}\{\mu_i\} = \sigma^2_i$
2	7.8	7.8	16.8	9.0	0.694
3	20.0	20.0	29.8	9.8	0.444
4	33.0	33.0	33.0	0.0	5.429
5	18.0	25.8	34.8	9.0	1.137
6	9.0	29.0	38.8	9.8	1.444
7	9.8	42.8	42.8	0.0	6.123



Critical Path (denoted by bold lines)

(d) The scheduled time of completing the project is 41.5 weeks. Therefore, the distance, in standard deviations, of the scheduled time ST from T_s, is given by

$$D_7 = \frac{ST - E(\mu_7)}{\sqrt{\text{var}(\mu_7)}} = \frac{41.5 - 42.8}{\sqrt{6.123}} - 0.52,$$

where $ST = 41.5$.

Therefore, we shall have

$$P\{z \le D_7\} = 0.30$$

which is the area under standard normal curve bounded by ordinates at $x = 0$ and $x = 0$ and $x = 0.52$.

The physical interpretation of this is that if the project is performed hundred times under the same conditions, then in 30 occasions it will take 41.5 weeks or less to complete the project. In other words, only 70 times the project will take more than 41.5 weeks for completion.

6.7 Questions

1. Define a "project". Discuss the guidelines for constructing a project network.
2. Define the following : (i) total float, (ii) free float, (iii) critical path.
3. Distinguish between PERT and CPM.
4. A construction company has listed down various activities that are involved in constructing a building. These are summarized along with predecessor (s) details in the following table :

Activity	Immediate Predecessor (s)
A	–
B	–
C	A
D	B
E	A, B
F	C, D
G	F, B
H	E, G
I	H, G
J	I, F

Activity	Immediate Predecessor (s)
K	J, L
L	A
M	K

Draw a project network for the above project.

5. Tasks $A, B, C, \ldots, H, I$ constitute a project. The notation $X < Y$ means that the task X must be finished before Y can begin. With this notation,

$$A < D, A < E, B < F, D < F, C < G, C < H, F < I, G < I.$$

Draw a graph to represent the sequence of tasks and find the minimum time of completion of the project, when the time (in days) of completion of each task is as follows :

(a)

Task :	A	B	C	D	E	F	G	H	I
Time :	8	10	8	10	16	17	18	14	9

(b)

Task :	A	B	C	D	E	F	G	H	I
Time :	5	9	14	4	3	10	19	12	10

6. Given the data below find the following :

(a) The expected task times and their variances.

(b) The earliest and latest expected times to reach each node.

(c) The variances of the earliest node times.

(d) The earliest and latest expected times to complete each task and the variances of the earliest times.

(e) The probabilities that each task will be completed on schedule.

Task	A	B	C	D	E	F
Least time	8	14	16	24	28	18.5
Greatest time	14	26	22	36	46	21.5
Most likely time	10	20	20	30	36	20
Scheduled completion time	20	20	40	80	80	100

Precedence relationships : A, B can start immediately : $A < C, D$; $B < C, D$; $C < E$; $D, E < F$.

Unit 7 ❑ Queuing Theory

Structure

7.1 Introduction

A queue or a waiting line is a common phenomenon in everyday life. For example, patients waiting at a doctor's clinic, persons waiting at a railway booking office, machines waithing for repair, ships waiting to be unloaded in the harbour and letters waiting to be typed all form queues. In queuing theory the individuals coming for service are called customers and those rendering service are called servers or service facilities.

The study of queues finds importance mainly in the fields of business (banks, supermarkets, booking offices), industries (production lines, storage, servicing of machines), engineering (communication networks, computers), transportation (airports, harbours, railways) and everyday life (elevators, restaurants, barber shops). The primary concern in such a study is to design and plan service facilities in such a way that congestion is minimized and the economic balance between the cost (or time) of service and the cost (or time) associated with waiting for service is maintained.

7.2 Queuing System

A queuing system comprises of customers arriving at a service station for certain service which is rendered by one or more servers. If an arriving customer finds all the servers busy he may wait in a queue. As soon as a customer is served, he departs from the system.

Customers arriving to the system may be classified as—(a) patient customers, and (b) impatient customers. A patient customer is one who decides to wait no matter how long the queue or the waiting time is. An impatient customer, on the other hand, gets dissuaded by the huge queue length or long waiting time. The different types of impatient customers are as follows—

(i) An arriving customer on seeing a queue may leave. Such a customer is said to balk.

(ii) An arriving customer may join the queue, but become impatient and leave after same time of waiting. Such a customer is said to renege.

(iii) If there be two or more parallel queues, a customer may shift from one queue to another which is moving faster. Such a customer is called a jokey.

7.3 Classification of a Queuing System

A queuing system is classified by the follows factors—

1. **The arrival (or input) process :** This defines the pattern of arrival of customers to a queuing system which is generally governed by some probabilistic law. Customers may arrive one by one or in fixed or variable batches. The latter is referred to as bulk arrival.

If the arrival process does not change with time it is called a stationary input process, else it is referred to as a transient input process.

An arrival process is said to be Markovian when arrivals occur in a Poisson fashion, i.e, customers arrive one at a time and the number of customers arriving per unit time has a Poisson distribution with mean arrival rate λ, which is independent of the customers already in the system. For such a process the inter-arrival time of customers has a one-parameter exponential distribution with mean $1/\lambda$.

2. **Qutput process :** The output process defines the mode of departure of customers from the system after being served. It is characterized by the service time

distribution for a service facility. The output process is said to be Markovian when the service time distribution is one-parameter exponential with mean service time $1/\mu$, independent of the number of customers in the system. Then the number of service completions per unit time follows a Poisson distribution with mean service rate μ.

3. **Queue discipline :** It is a rule by which customers are selected from a queue for rendering service. The most common queue discipline is "first come, first served" (FCFS) or "first in, first out" (FIFO) in which customers are served in order of their arrival. Other queue disciplines are "last come, first served" (LCFS) or "last in, first out" (LIFO) where the last arrival is served first, "selection in random order" (SIRO) where a customer is randomly selected from the queue for service, and selection by priority in which service preference is given to a cutsomer over some other customers in terms of certain factors like age, sex, importance, etc.

Under priority discipline service may be of two types—pre-emptive and non-pre-emptive. In pre-emptive service, when a high priority customer arrives the customer being served is removed and service is rendered to the high priority customer. On the other hand, in non-pre-emptive service, the high priority customer is given service only after service completion on the customer who was being served when he arrived.

4. **Service mechanism :** This refers to the number of server or service points to be maintained at a service station and their arrangement. The service points may be arranged in parallel or in series. In parallel arrangement, identical service is rendered by the different servers or service points. For such an arrangement, customers may wait in a single queue till a server becomes free as in a barber's shop or they may form separate queues in front of the different service points as in a supermarket. On the other hand, in a series arrangement a customer has to pass successively through all the service points before service is completed. Such a situation is often faced in public offices where different parts of a work are done at different service counters.

7.4 Kendall's Representation of Queuing Models

A queuing model is generally represented in the following symbolic form :

$$a/\,b/\,c/\,d/\,e,$$

where $a \equiv$ input (arrival) process

$b \equiv$ output process

$c \equiv$ number of servers in parallel

$d \equiv$ capacity of the system

$e \equiv$ queue discipline.

Representation by the first three symbols viz. *a/b/c* was introduced by Kendall. *a* and *b* may be any one of the following :

$M \equiv$ Markovian

$E_K \equiv$ K-Erlang

$G \equiv$ General

$D \equiv$ Deterministic

$SM \equiv$ Semi-Markovian

Thus, a queuing system M/M/2/∞/FCFS will mean the input and output processes are Markovian, the number of servers is 2, the system is of infinite capacity and the queue discipline is FCFS. Generally, if the system is of infinite capacity and FCFS is the queue discipline, the queuing system is reresented by only first three symbols, viz. *a/b/c*.

Notations : The following notations will be used in connection with a queuing system—

n = no. of customers in the system (i.e., no. of customers in the queue + no. of customers being served)

m = no. of customers waiting in the queue

c = no. of parallel service points

$P_n(t)$ = probability that there are n customers in the system at any time point t

P_n = steady-state (i.e., time independent) probability that there are n customers in the system.

(We have $p_n = \lim_{t \to \infty} P_n(t)$)

$E(n)$ = expected number of customers in the system in the steady state

$E(m)$ = expected number of customers waiting in the queue in the steady state

$E(v)$ = expected time spent by a customer in the system in system in steady state

$E(w)$ = expected waiting time of a customer in the queue in the steady state

$E(n)$, $E(m)$, $E(v)$ and $E(w)$ define the characteristics of a queuing system.

7.5 Queuing System with State Dependent Mean Arrival and Service Rates

Let λ_n = mean arrival rate to the system when there are n customers in the system

μ_n = mean service rate when there are n customers in the system

P_r [a new arrival in $(t, t + \Delta t)$ | n customers in the system at time t] = $\lambda_n \Delta t + 0(\Delta t)$ ... (1)

Pr [no new arrival in $(t, t + \Delta t)$ | n customers in the system at time t] = $1 - \lambda_n \Delta t + 0(\Delta t)$ ... (2)

Pr [a service completion in $(t, t + \Delta t)$ | n customers in the system at time t] = $\mu_n \Delta t + 0(\Delta t)$ ... (3)

Pr [no service completion in $(t, t + \Delta t)$ | n customers in the system at time t] = $1 - \mu_n \Delta t + 0(\Delta t)$ ... (4)

Pr [k new arrivals in $(t, t + \Delta t)$ | n customers in the system at time t] = $0(\Delta t)$, for $k \geq 2$ ... (5)

Pr [k service completions in $(t, t + \Delta t)$ | n customers in the system at time t] = $0(\Delta t)$, for $k \geq 2$ ... (6)

where $\dfrac{0(\Delta t)}{\Delta t} \to$ as $\Delta t \to 0$.

Then, since arrival and service completion are independent of one another,

$Pr\,(t + \Delta t)$ = Pr [n customers in the system at time $t + \Delta t$]

= Pr [n customers in the system at time t and no service completion and no new arrival in $(t, t + \Delta t)$] + $Pr[(n - 1)$ customers in the system at time t and 1 new arrival but no service completion in $(t, t + \Delta t)$] + $Pr[(n + 1)$ customers in the system at time t and 1 service completion but no arrival in $(t, t + \Delta t)$] + $\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} Pr[(n - k + s)$ customers in the system at time t, and k new arrivals and s service completions in $(t + \Delta t)$].

$$= Pn(t)\,[1 - \lambda_n \Delta t + 0(\Delta t)][1 - \mu_n \Delta t + 0(\Delta t)] + P_{n-1}(t)[\lambda_{n-1} \Delta t + 0(\Delta t)]$$
$$[1 - \mu_{n-1} \Delta t + 0(\Delta t)] + P_{n+1}(t)[1 - \lambda_{n+1} \Delta t + 0(\Delta t)][\mu_{n+1} \Delta t + 0(\Delta t)] + 0(\Delta t).$$

$= Pn(t)[1 - (\lambda_n + \mu_n)\Delta t] + P_{n-1}(t)\lambda_{n+1}\Delta t + P_{n+1}(t)\,\mu_{n+1}\Delta t + 0(\Delta t).$

For $n = 0$, noting that the question of service completion does not arise when there is no customer in the system, i.e., $\mu_o = 0$, and number of customers cannot be negative, we get

$$P_o(t + \Delta t) \quad = P_o(t)\,[1 - \lambda_o\Delta t] + P_1(t)\mu_1\Delta t + 0(\Delta t).$$

Hence.

$$\frac{d}{dt}P_0(t) = \lim_{\Delta t \to 0} \frac{e^{-\mu t'}(\mu t')^{n-1}}{(n-1)1} \sum_{n=0}^{\infty}\left[1 + \rho\int_0^t \mu e^{-\mu t(1-\rho)}dt'\right] = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

$$\frac{d}{dt}P_n(t) = \lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_0(t)}{\Delta t}$$

$$= -(\lambda_n + \mu_1)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{1+1}P_{n-1}(t),\ n \geq 1$$

These equations are called the differential difference equations. To get the steady state distribution of n we lake the limit as $t \to \infty$. Noting that $P_n(t) \to p_n$ and $P_n(t) \to \infty$..

we get from the above equations

$$0 = -\lambda_0 p_0 + \mu_1 p_1$$

$$0 = -(\lambda_n + \lambda_n)p_n + \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1},\ n \geq 1.$$

or, $\quad \lambda_0 p_0 = \mu_1 p_1$

$$(\lambda_n + \mu_n)p_n = \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1},\ n \geq 1.$$

Solving these equations we get

$$P_n = \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_0\mu_1\ldots\mu_{n-1}}\,p_0, n \geq 1 \qquad \ldots (1)$$

Since , $\sum_{n=0}^{\infty} p_n = 1$, we have

$$p_0 = \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_0\mu_1\ldots\mu_{n-1}}\right)^{-1}$$

Note : (1). For Markovian input and output processes the mean arrival rate and the mean service rate are given by λ and μ, respectively, which are independent of the number of customers in the system.

(2) For Markovian input and output processes (1) - (6) are satisfied.

Special Cases :

1. M/M/1/∞/FCFS (or simple M/M/1) Queuing System

Here $\lambda_n = \lambda,\ n > 0$

$\mu_n = \mu,\ n > 1$

We define $\rho = \frac{\lambda}{\mu}$, which is called the traffic intensity or utilization factor.

We assume that $\rho < 1$.

So the steady state distribution of the number of customers in the system is

$$P_n = \left(\frac{\lambda}{\mu}\right)^n p_0 = \rho^n \rho_0,\ n\ \ 0$$

$$\text{where } p_0 = \left(1 + \sum_{n=1}^{\infty}\left(\frac{\lambda}{\eta}\right)^n\right)^{-1} = \left(\sum_{n=1}^{\infty}\rho^n\right)^{-1}$$

$$= \left(\frac{1}{1-\rho}\right)^{-1},\ \text{since } \rho < 1$$

$$= 1 - \rho.$$

Hence

$$p_n = (1-\rho)\rho^n,\ n \geq 1.$$

Therefore,

(i) $$E(n) = \sum_{n=0}^{\infty} np_n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = \frac{(1-\rho)\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

(ii) $$E(m) = \sum_{n=1}^{\infty}(n-1)p_n,\ \text{since } n = m + 1 \text{ as there is only one server}$$

$$= \sum_{n=1}^{\infty} np_n - \sum_{n=1}^{\infty} p_n$$

$$= \frac{\rho}{1-\rho} - \rho$$

$$= \frac{\rho^2}{1-\rho}$$

(iii) $\text{Var}(n) = \sum_{n=1}^{\infty} (n - E(n))^2 p_n,$

$$= \sum_{n=1}^{\infty} n^2 p_n - \{E(n)\}^2$$

$$= (1-\rho) \sum_{n=1}^{\infty} n^2 p^n - \frac{\rho^2}{(1-\rho)^2}$$

$$= \frac{(1-\rho)\rho(1+\rho)}{(1-\rho)^3} - \frac{\rho^2}{(1-\rho)^2}$$

$$= \frac{\rho}{(1-\rho)^2}.$$

(iv) To find $E(v)$ and $E(w)$, we first find the waiting time distribution of an arriving customer in the queue.

Let $\varphi_w(t)$ denote the c.d.f. of the time spent by a customer in the queue $w \geq 0$. Since an arriving customer does not have to wait for service when there are no customer in the system, we get

$$\begin{aligned}\varphi_w(0) &= P_r[w = 0] \\ &= P_r\text{ [no customer in the system]} \\ &= p_0 \\ &= 1 - \rho.\end{aligned}$$

When a new arrival finds n customers in the system, $n \geq 1$, he has to wait in the queue till all the n customers are served. Hence, for $t > 0$,

$Pr\ [t \leq w \leq t + dt \mid n$ customers in the system when the new customer arrives]

$= Pr\ [(n-1)$ customers are served in time $t] \times Pr$ [one customer is served in time dt]

$$= \frac{e^{-\mu t}(\mu t)^{n-1}}{(n-1)1}\mu dt = \psi_n(t)dt, \text{ say.},$$

since the number of customers served in time t follows a Poisson distribution with mean μt.

Therefore, for $t > 0$,

$$\varphi_w(t) = Pr[w \le t]$$

$$- \varphi_w(0) + \sum_{n=1}^{\infty} \int_0^t p_n \psi_n(t) dt$$

$$= (1 - \rho) + \sum_{n=1}^{\infty} \int_0^t (1 - \rho)^n \frac{e^{-\mu t'} (\mu t')^{n-1}}{(n-1)1} \mu dt'$$

$$= (1 - \rho) + \left[1 + \rho \int_0^t \mu e^{-\mu t'} \left(\sum_{n=1}^{\infty} \frac{(\mu t' \rho)^{n-1}}{(n-1)!} \right) dt' \right]$$

$$= (1 - \rho) + \left[1 + \rho \int_0^t \mu e^{-\mu t'} \left(\sum_{n=0}^{\infty} \frac{(\mu t' \rho)^{n}}{(n-1)!} \right) dt' \right]$$

$$= (1 - \rho) + \left[1 + \rho \int_0^t \mu e^{-\mu t (1-\rho)} dt' \right]$$

$$= (1 - \rho) + \left[1 + \rho \left\{ \frac{1 - e^{-\mu(1-\rho)t}}{1-\rho} \right\} \right]$$

$$= - 1 - \rho e^{-\mu(1-\rho)t}.$$

So the distribution of waiting time in queue is

$$\varphi_w(t) = \begin{cases} 1-\rho, & t = 0 \\ 1-\rho e^{-\mu(1-\rho)t}, & t > 0 \end{cases}$$

Hence,

$$E(w) = \int_0^{\infty} t d\varphi_w(t)$$

$$= \int_0^{\infty} \rho\mu(1 - \rho) e^{-\mu(1-\rho)t} dt$$

$$= \frac{\rho}{\mu(1-\rho)} \int_0^{\infty} x e^{-x} dx, \text{ putting } x - \mu(1 - \rho)$$

$$= \frac{\rho}{\mu(1-\rho)}$$

Since, $v = w$ + service time,

$$E(v) = E(w) + \text{mean service time}$$

$$= \frac{\rho}{\rho(1-\rho)} + \frac{1}{\mu}$$

$$= \frac{1}{\rho(1-\rho)}$$

$E(w)$ and $E(v)$ may also be found using Little's Formula. By this formula we have

$$E(v) \cong \frac{E(n)}{\text{mean arrival rate}}$$

$$E(w) \cong \frac{E(m)}{\text{mean arrival rate}}$$

In case of M/M/I queuing system, the formula gives exact expressions for $E(v)$ and $E(w)$.

Example 1 : A T.V. repairman finds that the time on his jobs has an exponential distribution with mean 30 mins. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8 hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in? What is the average time spent by such a set in the queue and in the system?

Solution : It is given that

$$\lambda = \frac{10}{8} = \frac{5}{4} \text{ sets per hour}$$

$$\mu = \frac{1}{30} \times 60 = 2 \text{ sets per hour}$$

$\therefore \quad \rho \quad$ traffic intensity $= \frac{\lambda}{\mu} = \frac{5}{8}$

$\therefore \quad \rho_0 = 1 - \rho = \frac{3}{8}$.

Hence, idle time for a repairman in 8-hour day

$$= \frac{3}{8} \times 8 = 3 \text{ hours.}$$

$$E(n) = \frac{\rho}{1-\rho} = \frac{5}{3} = 1\frac{2}{3} \text{ jobs.}$$

$$E(v) = \frac{1}{\mu(1-\rho)} = \frac{1}{2 \times 3/8} = \frac{4}{3} = 1\frac{1}{3} \text{ hours.}$$

$$E(w) - E(v) - \frac{1}{\mu} - \frac{4}{3} - \frac{1}{2} - \frac{5}{6} \text{ hour.}$$

2. **M/M/C/∞/FCFS (or M/M/2) Queuing System :**

Here

$$\lambda_n = \lambda \text{ for all } n \geq 0.$$

Since the service time distribution is exponent with mean $1/\mu$ for each of the C servers (i.e., mean service rate is μ for each server), and for $n \leq C$, all the customers in the system can be served simultaneously while for $n > C$, only C customers can be served at a time, we have

$$\mu_n = n\mu \text{ if } n \leq C$$

$$= C\mu \text{ if } n > C.$$

For this model, the traffic intensity is given by $\frac{\lambda}{C\mu}$ which is assumed to be < 1.

From (1) we get the steady-state distribution of the number of customers in this system as

$$p_n = \frac{\rho^n}{n!}, p_0 \text{ if } n \leq C, \text{ where } \rho = \frac{\lambda}{\mu}$$

$$= \frac{\rho^n}{C!C^{n-c}} p_0, \text{ if } n > C$$

$$\text{where } p_0 = \left(1 + \sum_{n=1}^{C-1} \frac{\rho^n}{n!}\frac{\lambda}{\mu} + \sum_{n=c}^{\infty} \frac{\rho^n}{C!C^{n-c}}\right)^{-1} - \left(\sum_{n-0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{C!}\sum_{n-0}^{\infty}\left(\frac{\rho}{C}\right)^n\right)^{-1}$$

$$= \left(\sum_{n-0}^{C-1} \frac{\rho^n}{n!} + \frac{\rho^C}{(C-1)!(C-\rho)}\right)^{-1}$$

Therefore,

(i) $E(m) = \sum_{n=c}^{\infty} (n-C)p_n$, since a queue is formed for $n \geq C$ and no. of customers in the queue is $n - C$.

$$= \sum_{n-c}^{\infty} (n-C)\frac{\rho^n}{C!C^{n-C}} p_0$$

$$= p_0 \frac{\rho^C}{C!} \sum_{n-0}^{\infty} n\left(\frac{\rho}{C}\right)^n, \text{ replacing } n-C \text{ by } n$$

$$= p_0 \frac{\rho^C}{C!} \cdot \frac{\rho}{C!} \cdot \frac{C^2}{(C-\rho)^2} = p_0 \frac{\rho^{C+1}}{(C-1)!(C-\rho)^2}$$

(ii) $E(n)$ = expected queue length + expected number of customers arriving during a service time $- E(m) + \dfrac{\lambda}{\mu}$.

(iii) By Little's formula,

$$E(v) \cong \frac{E(n)}{\lambda}$$

(iv) $E(w)$ = expected time spent by a customer in the system —expected service time

$$= E(v) - \frac{1}{\mu}$$

(v) Expected no of idle servers $= (C-1)p_1 + (C-2)p_2 + ... + 1p_{c-1}$

$$= \sum_{x=1}^{C-1} (C-x)p_x.$$

(vi) Efficiency of the queuing model

$$= \frac{\text{Expected no. of customers being served}}{\text{Total no. of servers}}$$

$$= 1 - \sum_{x=1}^{C-1} (C-x)P_x / C.$$

Example 2 : A super market has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if people arrive in a Poisson fashion at the rate of 10 an hour.

(a) What is the probability of having to wait for serviec?

(b) Find the average queue length and the average number of customers in the system.

(d) How long does an arriving customer have to wait for service?

Solution : This is a *M/M/C* queuing model with $C = 2$

$$\lambda - \frac{10}{60} - \frac{1}{6} \text{ per min.}$$

$$\mu - \frac{1}{4} \text{ per minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{1/6}{1/4} = \frac{2}{3}.$$

Therefore,

$$p_0 = \left\{ \sum_{n=0}^{1} \frac{\left(\frac{2}{3}\right)^n}{n!} + \frac{\left(\frac{2}{3}\right)^2}{1!\left(2 - \frac{2}{3}\right)} \right\}^{-1}$$

$$= \left\{1 + \frac{2}{3} + \frac{1}{3}\right\}^{-1} = \frac{1}{2}.$$

$$p_1 = \rho p_0 = 1/3.$$

(a) $Pr\,[n \geq 2] - 1 - p_0 - p_1 - \frac{1}{6} - 0.167.$

(b) Expected no of idle girls $= 2p_0 + 1p_1 = 4/3$.

$\therefore$ Probability a girl is idle $= \dfrac{\text{expected no. of idle girls}}{\text{total no. of girls}} = \dfrac{4/3}{2} - 2/3 - 0.67.$

Hence the expected percentage of idle time for each girl is 67%.

(c) $$E(m) = p_0 \frac{\rho^{C+1}}{(C-1)!(C-\rho)^2} = \frac{1}{2} \frac{\left(\frac{2}{3}\right)^3}{1!\left(\frac{4}{3}\right)^2} = \frac{4}{3} = 1\frac{1}{3}.$$

$$E(n) = E(m) + \rho.$$

$$= \frac{4}{3} + \frac{2}{3} - 2.$$

(d) $E(w) \cong \frac{E(m)}{\lambda} = \frac{4/3}{1/6} = 8$ minutes.

3. M/M/C/N/FCFS Queuing System

Here we assume $C \leq N$, since if $C > N$ there will be no queue.

Since no customer can enter the system where there are N customers already in the system, we have

$$\lambda_n = \lambda \text{ if } n < N$$
$$= 0 \text{ if } n \geq N.$$

And,

$$\mu_n = n\mu \text{ if } n \leq N$$
$$= C\mu \text{ if } n > N.$$

So, the sdteady = – state distribution of the number of customers in the system is

$$P_n = \frac{\rho^n}{n!} p_0 \text{ if } n \leq C, \text{ where } \rho = \frac{\lambda}{\mu}$$

$$= \frac{\rho^n}{C!C^{n-C}} p_0 \text{ if } C < n < N$$

$$= 0 \text{ if } n > N.$$

where

$$p_0 - \left(\sum_{n=0}^{C-1} \frac{\rho^n}{n!} + \frac{1}{C!} \sum_{n=C}^{N} \frac{\rho^n}{C^{n-C}} \right)^{-1}$$

Then,

(i) $E(m) = \sum_{n=C}^{N} (n - C) p_n$

$$= p_0 \frac{\rho^C}{C!} \sum_{n=C}^{N} (n - C) \frac{\rho^{n-C}}{C^{n-C}}$$

$$= p_0 \frac{\rho^C}{C!} \sum_{n=0}^{N-C} n\left(\frac{\rho}{C}\right)^n$$

$$= p_0 \frac{\rho^{C+1}}{C!} \sum_{n=0}^{N-C} n\left(\frac{\rho}{C}\right)^{n-1}$$

$$= p_0 \frac{\rho^{C+1}}{C!} \sum_{n=0}^{N-C} \frac{d}{d\rho'}(\rho')^n, \text{ where } \rho' = \frac{\rho}{C} < 1$$

$$= p_0 \frac{\rho^{C+1}}{C!} \frac{d}{d\rho'}\left(\sum_{n=0}^{N-C} \rho'^n\right)$$

$$= p_0 \frac{\rho^{C+1}}{C!} \frac{d}{d\rho'}\left(\frac{1-\rho^{N-C+1}}{1-\rho'}\right)$$

$$= p_0 \frac{\rho^{C+1}}{C!}\left(\frac{1-\rho'^{N-C+1}-(1-\rho')(N-C+1)\rho'^{N-C}}{(1-\rho')^2}\right).$$

(ii) $E(n) = \sum_{n=C}^{N} np_n$

$$= \sum_{n=C}^{N} np_n + E(m) + C\sum_{n=C}^{N} p_n$$

$$= E(m) + C\sum_{n=0}^{C-1} (n-C)p_n$$

$$= E(m) + C - p_0 \sum_{n=0}^{C-1} (C-n)\frac{\rho^n}{n!}.$$

By Little's Formula,

(iii) $E(v) \cong \dfrac{E(n)}{\lambda'}$,

where $\lambda' = \lambda(1 - p_N)$ = mean number of customers arriving per unit time,

(iv) $E(w) \cong \dfrac{E(m)}{\lambda'}$

or, $E(w) = E(v) - 1/\mu$.

Notes : (a) For $C = 1, N = \infty$, we get the M/M/1 queuing system.

(b) For $N = \infty$, we get the M/M/C queuing system.

Example 3. In an automobile inspection station there are three inspection states. Cars wait in such a way that when a stall becomes vacate the car at the head of the line pulls up to it. The station can accommodate at most four cars waiting at a time. The arrival pattern is Poisson with mean one car every minute during the peak hours. The service time is exponential with mean 6 minutes. Find the average number of customers in the system during peak hours, the average waiting time and the average number per hour that cannot enter the station because of full capacity.

Solution : Here $\lambda = 1$ per minute

$$\mu - \frac{1}{6} \text{ per minute}$$

$$C = 3, N = 7.$$

Therefore, $\rho - \frac{\lambda}{\mu} = 6$

$$p_0 = \left(\sum_{n=0}^{2} \frac{\rho^n}{n!} + \frac{1}{C!}\rho^3 \frac{1-(\rho / C)^5}{1-\rho / C} \right)^{-1}$$

$$= \frac{1}{1141}$$

$\therefore$ $E(m) = 3.09$ cars

$$E(n) = E(m) + C - p_0 \sum_{n=0}^{C-1} (C-n)\frac{\rho^n}{n!} = 6.06 \text{ cars.}$$

$$E(v) \cong \frac{E(n)}{\lambda(1-p_7)} = \frac{6.06}{1-\dfrac{6^7}{1141 \times 3^4 \times (3!)}} = 12.3 \text{ mins.}$$

Expected no. of cars per hour that cannot enter the station is

$$60\lambda p_7 = 30.4 \text{ cars per hour.}$$

7.6 Non-Poisson queuing systems

Queuing models in which the arrival and / or departure patterns may not be Poisson are referred to as non-Poisson queuing models.

$M/E_k/1/\infty$/FCFS (or $M/E_k/1$) queuing system.

In this model the service time has a k-Erlang distribution given by the p.d.f.

$$f(t) = \frac{(k\mu)^k t^{k-1} e^{-k\mu t}}{k!}, \; t > 0, \, \mu > 0, \, k > 0.$$

The above distribution has mean $1/\mu$.

A k-Earlang distribution, for k a positive integer, can be looked upon as the distribution of the sum of k independent exponential variables each with mean $1/k\mu$. Thus, service cand be thought of as occurring in k phases, where the service time in each phase has an exponential distribution with mean $1/k\mu$. This consideration is conveniently used to analyze the $M/E_k/1$ model.

Let n denote the number of service phases in the system. Each arrival adds n phases of service to the system and each departure removes n service phases from the system. Hence, if

$$P_n(t) = Pr\ [n \text{ service phases in the system at time } t],$$

Then the difference equations are as follows—

$$\begin{aligned} P_n(t + \Delta t) = {} & P_n(t)[1 - \lambda\Delta t + 0(\Delta t)][1 - k\mu\Delta t + 0(\Delta t)] \\ & + P_n(t)[\lambda\Delta t][k\mu\Delta t] + P_{n+1}(t).[1 - \lambda\Delta t + 0(\Delta t][k\mu\Delta t] \\ & + P_{n-k}(t).[\lambda\Delta t][1 - k\mu\Delta t + 0(\Delta t)], \; n \geq 1. \end{aligned}$$

And $P_0(t + \Delta t) = P_0(t)[1 - \lambda\Delta t + 0(\Delta t)] + P_1(t)[k\mu\Delta t], \; n = 0.$

Here a negative subscript indicates the term is zero.

Following the usual procedure of obtaining the differential-difference equations and then the steady-state difference equations, we have, writing the steady-state distribution as $\{P_n\}$,

$$0 = -(\lambda + k\mu)P_n + k\mu P_{n+1} + \lambda P_{n-k;} \quad n \geq 1.$$

and $\quad 0 - \lambda P_0 + k\mu P_1.$

Letting $\quad [\lambda/k\mu] = \rho$, we have

$$(1+\rho)P_n = \rho P_{n-k} + P_{n+1}, \quad n \geq 1, \qquad \dots (1)$$

and $$\rho P_0 = P_1 \qquad \dots (2)$$

To solve these difference equations, we make use of the generating function defined by

$$P(x) = \sum_{n=0}^{\infty} P_n x^n \qquad \dots (3)$$

Multiplying (1) by x^n and then taking summation over n from 0 to ∞ and using (2) and (3), we get

$$(1+\rho)P(x) - P_0 = \rho x^k P(x) + \frac{1}{x}[P(x) - P_0].$$

This gives after simplification,

$$P(x) = P_0\left[1 - \rho x\frac{(1-x^k)}{1-k}\right]^{-1}$$

$$= P_0 \sum_{n=0}^{\infty} (\rho x)^n \left[1 - \rho x\frac{(1-x^k)}{1-k}\right]^n$$

since $(1-z)^{-1} = 1 + x + x^2 + \dots\dots$

$$= P_0 \sum_{n=0}^{\infty} \rho^n (x + x^2 + x^3 + \dots + x^k)^n.$$

Now, for $x = 1$

$$P(1) = P_0 \sum_{n=0}^{\infty} \rho^n k^n$$

or, $$1 = P_0\left[\frac{1}{1-k\rho}\right], \text{ since } P(1) = 1$$

or, $$P_0 = 1 - k\rho.$$

$$\therefore \quad P(x) = (1-k\rho)\sum_{n=0}^{\infty} (\rho x)^n (1-x^k)^n (1-x)^{-n}.$$

But, $(1-x^k)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x^k)^i$

and $(1-x)^n = \sum_{i=0}^{n} (-1)^j \binom{-n}{j} x^j$

where $\binom{-n}{j} = (-1)^j \binom{n+j-1}{j}$

Therefore,

$$P(x) = (1-k\rho) \sum_{n=0}^{\infty} (x\rho)^m \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x^k)^i \sum_{j=0}^{\infty} \binom{m+j-1}{j} x^j$$

$$= (1-k\rho) \sum_{m=0}^{\infty} \rho^m \sum_{j=0}^{\infty} \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{m+j-1}{j} x^{j+ik+m}$$

Hence, $\quad P_n$ = Coefficient of x^n in $P(x)$

$$= (1-k\rho) \sum_{i,j,k} \rho^m (-1)^i \binom{m}{i} \binom{m+j-1}{j} \text{ for } j+ik+m=n.$$

Characteristics of the model

(i) Average number of phases in the system $E(n_p)$ is obtained as follows :

Since the system of difference equations is

$$(1+\rho)P_n = \rho P_{n-k} + P_{n+1}, \quad n \geq 1$$

therefore, $\quad (1-\rho) \sum_{n=1}^{\infty} n^2 P_n = \rho \sum_{n=k}^{\infty} n^2 P_{n-k} + \sum_{n=1}^{\infty} n^2 P_{n+1}$

$$= \rho \sum_{x=0}^{\infty} (x+k)^2 P_x + \sum_{y=2}^{\infty} (y-1)^2 P_y \qquad \text{(where } n-k=x \text{ and } n+1=y\text{)}$$

$$= \rho \sum_{x=k}^{\infty} (x^2+2xk+k^2) P_x + \sum_{y=1}^{\infty} (y^2-2y+1) P_y$$

$$\left(\text{since } \sum_{y=2}^{\infty} (y-1)^2 P_y = \sum_{y=1}^{\infty} (y-1)^2 P_y\right)$$

$$= (1+\rho) \sum_{n=1}^{\infty} n^2 P_n + \rho \sum_{n=0}^{\infty} (2nk+k^2) P_n + \sum_{n=1}^{\infty} (-2n+1) P_n$$

or, $\quad 0 = \rho\left[2k \sum_{n=0}^{\infty} np_n + k^2 \sum_{n=0}^{\infty} p_n\right] + \sum_{n=1}^{\infty} P_n - 2\sum_{n=1}^{\infty} nP_n.$

or, $2(1-k\rho)\sum_{n=0}^{\infty} n.P_n = \rho k^2 - P_0 + 1,$

since $\sum_{n=0}^{\infty} P_n = 1$ and $\sum_{n=1}^{\infty} nP_n = \sum_{n=0}^{\infty} nP_n$

$$\therefore \quad E(n_p) = \frac{\rho k^2 + 1 - (1-k\rho)}{2(1-k\rho)}$$

$$= \frac{k(k+1)\rho}{2(1-k\rho)}$$

i.e., $E(n_\rho) - \frac{k(k+1)}{2} \cdot \frac{\lambda / k\mu}{1 - k\lambda / k\mu} = \left(\frac{k+1}{2}\right)\frac{\lambda}{\mu-\lambda}.$

(ii) Average waiting time of the phases in the system is given by

$$E(w_p) = \frac{E(n_p)}{\mu}$$

$$= \left(\frac{k+1}{2\mu}\right)\frac{\lambda}{\mu-\lambda}$$

(iii) Average waiting time of an arrival is given by

$$E(w) = \frac{E(w_p)}{k}$$

$$= \left(\frac{k+1}{2k}\right)\frac{\lambda}{\mu(\mu-\lambda)}$$

(iv) Average number of units in the system is given by

$$E(n) = \lambda E(v) = \lambda[E(w) + \frac{1}{\mu}]$$

$$= \left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}.$$

(v) Average queue length is given by

$$E(m) = E(n) - \frac{\lambda}{\mu}$$

$$= \left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu(\mu-\lambda)}.$$

(vi) Average time an arrival spends in the system is given by

$$E(v) = E(w) + \frac{1}{\mu}$$

$$= \left(\frac{k+1}{2k}\right)\frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}.$$

Particular case : ($M/E_k/1/1/FCFS$). This model differs from $M/E_k/1$ model in the sense that in this case the capacity of the system is unity, i.e., there is no queue and the system contains only one customer.

Let the customer be in n-th phase of service, where

$$1 < n < k.$$

Like in $M/E_k/1$, it can be easily seen that the following set of steady-state difference equations govern this model :

$$0 = -k\mu P_n + k\mu P_{n+1}, \qquad 1 \le n < k$$

$$0 = -k\mu P_k + \lambda P_0, \qquad n = k$$

$$0 = -\lambda P_0 + k\mu P_1, \qquad n = 0$$

These equations provide us the following relations :

$$P_1 = \frac{\lambda}{k\mu} P_0$$

$$P_k = \frac{\lambda}{k\mu} P_0$$

and $P_{n+1} = P_n$ for $1 < n < k$.

$\therefore$ For $n = 1, 2, ..., k-1$, we see that

$$P_1 = P_2 = P_3 = ... - P_{k-1} = P_k.$$

Thus, $P_1 - \frac{\lambda}{k\mu} P_0$ for $1, 2, ..., k$.

To obtain the value of P_0, we make use of the fact that,

$$\sum_{i=0}^{k} P_i = 1.$$

$$\text{or,}\quad P_0 + \sum_{i=0}^{k} P_i = 1$$

$$\text{or,}\quad P_0 + \sum_{i=2}^{k} \frac{\lambda}{k\mu} P_0 = 1$$

$$\text{or,}\quad P_0 = \left[1 + \frac{\lambda}{k\mu} \sum_{i=1}^{k} \frac{1}{k}\right]^{-1}$$

$$= \left(1 + \frac{\lambda}{k\mu}\right)^{-1}.$$

$$\text{Hence } P_n = \frac{1}{k} \cdot \frac{\lambda}{k+\mu}.$$

Example : A hospital clinic has a doctor examining every patient brought in for a general check-up. The doctor averages 4 minutes on each phase of the check-up although the distribution of time spent on each phase is approximately exponential. If each patient goes through four phases in the check-up and if the arrivals of the patients to the doctor's office are approximately Poisson at the average rate of three per hour, what is the average time spent by a patient waiting in the doctor's office? What is the average time spent in the examination? What is the most probable time spent in the examination?

Solution : We are given

$k = 4$, mean arrival rate = 3 patients per hour

i.e., $\lambda = 3$ per hour.

Service time per phase $= \dfrac{1}{4\mu} = 4$ minutes.

$$\therefore\quad \mu = \frac{1}{4\times 4} = \frac{1}{16} \text{ patients per minute}$$

$$\therefore\quad E(w) = \left(\frac{4+1}{4\times 4}\right) - \frac{3}{\frac{15}{4}\left(\frac{15}{4} - 3\right)}$$

$$= 40 \text{ minutes.}$$

Average time spent in the examination = $\frac{1}{\mu}$ – 16 minutes.

Most probable time spent in the examination (i.e., mode of the service time distribution)

$$= \frac{k-1}{k\mu}$$

$$= \frac{4-1}{4\times\frac{1}{16}} = \frac{3}{1/4} = 12 \text{ minutes.}$$

M/G/1 queuing system : Here the arrival process is Poisson with mean rate λ, say while the service time has a general distribution with mean $\frac{1}{\mu}$ and variance σ^2, say. Below are the characteristics of the model :

$$E(n) = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} + \rho, \text{ where } \frac{\lambda}{\mu} = \rho < 1.$$

This is known as Pollaczek-Khinchin Mean Value formula.

$$E(m) = E(n) - \rho$$

$$= \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$$

$$E(w) \cong \frac{E(m)}{\lambda} = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)}$$

$$E(v) \cong \frac{E(n)}{\lambda} = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)} + \frac{1}{\mu}$$

Note : For exponential service time distribution with mean $1/\mu$, we have $\sigma^2 = 1/\mu^2$.

Using this value of σ^2 we can obtain $E(n)$, $E(m)$, $E(w)$ and $E(v)$ for *M/M/*1 queuing system.

7.7 Questions

1. At what average rate must a clerk at a supermarket work in order to ensure a probability of 0.90 that the customer will not have to wait longer than 12 minutes? It is assumed that there is only one server to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

2. At a one-man barber shop customers arrive according to Poisson process with mean arrival rate of 5 per hour and his hair cutting time is exponentially distributed with an average haircut taking 10 minutes. It is assumed that because of his excellent reputation customers were always willing to wait. Calculate

   (i) the average number of customers in the shop and the average number of customers waiting for haircut;

   (ii) the percentage of time an arrival can walk right up to the barber's chair without having to wait;

   (iii) the percentage of customers who have to wait prior to getting into the barber's chair.

3. A telephone exchange has two long distance operators. The telephone company finds that during the peak load, long distance calls arrive on these calls is approximately exponentially distributed with mean length 5 minutes.

   (a) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?

   (b) If the subscribers will wait and are serviced in turn, what is the expected waiting time?

4. At a port, let there be six unloading berths and four unloading crews. When all the berths are full, arriving ships are diverted to an overflow facility 20 miles down the river. Tankers arrive according to a Poisson process with a mean of one every 2 hours. It takes an unloading crew, on the average, ten hours to unload a tanker, the unloading time following an exponential distribution. Find

   (a) On the average, how many tankers are at the port?

   (b) On the average, how long does a tanker spend at the port?

   (c) What is the average arrival rate at the overflow facility?

5. In a certain bank the customers arrive according to a Poisson distribution with mean of 4 per hour. From observation on the teller's performance, the mean service time is estimated to be 10 minutes with a standard deviation of 5 minutes. It is felt that the Erlang would be a reasonable assumption for the distribution of the teller's service time. Also, there is no limit on the number of customers entering the bank. Find (a) how long a customer has to wait, on an average, to get service, (b) the average number of customers waiting for service.

Unit 8 ❑ Inventory Control

Structure

8.1 Introduction

Inventory is any stock of goods that is maintained for the smooth and efficient running of a business. For example, a business may have as inventory of raw material, semi-finished goods or finished goods. It should be noted that any stock that is being used to meet demand is not inventory. Inventory is an 'idle' stock which is stored in order to meet future demand. Further, items to be stocked as inventory must be of some economic value. Thus, Fred Hansmann right fully defined inventory as 'an idle resource of any kind provided that such resource has economic value".

8.2 Inventory Decisions

For every item to be stored as inventory, the two main decisions to be taken by the inventory manager are—(a) how much to order, and (b) when to order. In order to make his decisions the manager has to take into account the different costs that are associated with procuring and maintaining inventory. The decisions taken are such that the total (expected) cost incurred is minimized, or the total (expected) profit is maximized.

8.3 Costs Involved in Inventory Problems

The costs involved in inventory problems may be broadly classified as follows :

I. **Procurement Cost :** This has the following components—

(i)	*Ordering cost.*	This is the cost of placing at order. It may be dependent or independent of the order quantity.
(ii)	*Set-up cost.*	This is the cost of setting up the production process when items to be stored are produced and not procured from an outer source.
(iii)	*Purchase cost.*	Purchase cost per item may be affected by the order quantity owing to quantity discounts.

II. **Holding or Carrying Cost :** The holding or carrying cost, which is associated with holding goods in stock, is usually assumed to vary directly with the size of the inventory well as the time for which an item remains in inventory. The different components of this cost are as follows—

(i)	*Storage cost.*	This involves the rent of storage space for inventory or depreciation and interest even if one's own space is used.
(ii)	*Insurance cost.*	This is the cost of insuring inventory against possible damage.
(iii)	*Handling cost.*	This is the loss associated with wear-and-tear owing to repeated handling of stock.
(iv)	*Deterioration cost.*	This involves the loss due to deterioration or decay of items during storage.
(v)	*Obsolescence cost.*	Such a cost arises due to the items in stock going out of fashion.

III. **Shortage Cost :** The penalty cost that is incurred as a result of runing out of stock (i.e., shortage) known as shortage or stock-out cost. In the case where the unfilled demand can be satisfied at a later date (i.e., backlog case), this costs is generally assumed to very directly both with the shortage amount and delay time. On the other hand, when the unfilled demand is lost (i.e., lost sales case), shortage cost is proportional to the shortage quantity only.

An indirect shortage cost is the cost of a lost goodwill owing to the inability meet demand.

IV. **Excess Cost :** For items which can be used during a specified period only lil newspapers, if the stock on hand is less than the demand, the excess stock may be of r value or may be possible to dispose off at a reduced cost known as salvage value. Tl excess cost is the loss defined by the difference between the normal value and tl salvage value.

8.4 Classification of Inventory Models

The different bases for classification of inventory models are as follows—

(i) *Nature of knowledge about demand.* When demand is completely knov the inventory model is called a certain or deterministic model. On the oth hand, if demand be a random variab the model is known as a risk model.

(ii) *Number of orders that are placed* If only one order be placed during tl planning period or planning horizo (i.e., the time period over which tl model is implemented), we call tl model a static model. A dynamic mod is one in which two or more orders a placed during the planning period.

(iii) *Lead time.* Lead time is the time lag betwee placing an order and its delivery supply. Lead time may be zero, whic is in the case of shelf items, a consta or a random variable.

8.5 Different Inventory Models

I. Deterministic Inventory Models

Model 1 : Dynamic Certainty (or Deterministic) Model with no shortage and zero lead time.

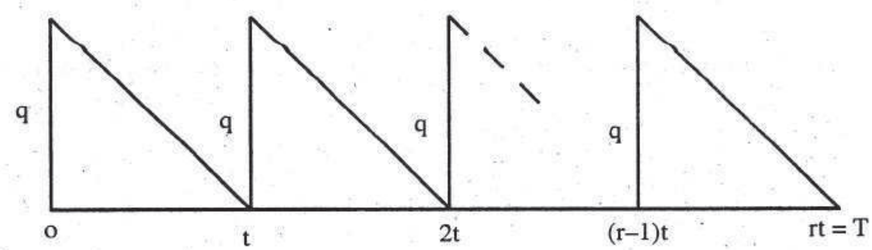
Let D be the known demand during a given planning horizon (O, T). Suppose withdrawal from inventory is made at an uniform rate $R = D/T$ and shortage is not allowed. Further, lead time is assumed to be zero, which means that supply will be instantaneous on ordering.

Let

C_s = ordering cost per order, independent of the order quantity

C_1 = carrying cost per unit quantity per unit time.

The inventory policy is to place an order for q units whenever the stock on hand comes down to zero. Since demand occurs at a constant rate, the policy implies that orders be placed at equidistant time points $o, t, 2t, ..., (r-1)t$ as indicated in the following figure.



Here r, called the order frequency, is given by $r = \frac{T}{t}$. Also, $r = \frac{D}{q}$, since shortages are not allowed.

The inventory manager has to decide upon q and t. Since $r = \frac{T}{t} = \frac{D}{q}$, we can write $t = \frac{Tq}{D}$. Thus the number of independent variables will be one. Let it be q. We now find q so as to minimize the total cost over $(0, T)$.

Let

$C(q)$ = total cost over $(0, T)$

$= \text{total ordering cost over } (0, T) + \text{total carrying cost over } (0, T).$

The total ordering cost over (0, T) is given by the number of orders placed in (0 T) × ordering cost per order $= rC_S = \frac{D}{q} C_S$.

The total inventory carrying cost over (0, T) is determined as follows—

The average inventory over (0, T) $= \frac{1}{2}$ (maximum level + minimum level) $= \frac{q}{2}$.

This is also the average inventory over each reorder interval, i.e., each interval of length t. So, the inventory carrying cost over (0, T).

$$= C_1 \times \text{average inventory} \times \text{time for which inventory is carried}$$

$$= C_1 \frac{q}{2} T.$$

$$\therefore \quad C(q) = \frac{D}{q} C_S + C_1 \frac{q}{2} T.$$

This is a strictly convex function of q since $\frac{d^2}{dq^2} C(q) > 0$.

Hence optimal q which minimizes $C(q)$ is the unique solution to $\frac{d}{dq} C(q) = 0$, which gives

$$q_{opt} = \sqrt{\frac{2DC_S}{GT}}.$$

This is known as the square-root formula. q_{opt} is called the econmic order quantiy (*EOQ*) or economic lot size (*ELS*) as it minimizes the total cost.

The minimum cost is, therefore, given by

$$C(q_{opt}) = \sqrt{2DC_S C_1 T}$$

The optimal reorder interval is

$$t_{opt} = \frac{Tq_{opt}}{D}$$

Example : Novelty Ltd. carries a wide assortment of items for its customers. One item, in particular, is very popular. Desirous of keeping its inventory under control, a decision is taken no order only the optimal economic quantity, for this item, each time.

You have the following information. Make your recommendations :

Annual Demand : 16,00,00 units, Carrying cost : Re. 1 per unit, Cost per order : Rs. 50.

Determine the optimal economic quantity.

Solution : Here $T = 1$ year;

$D = 1{,}60{,}000$ units

$C_S =$ Rs. 50 per order

$C_1 =$ Re. 1 per unit per year

So, $q_{opt} = \sqrt{\frac{2DC_s}{C_1T}}$

$= 4000$ units

$t_{opt} = .0025$ year

and minimum cost is

$C(q_{opt}) =$ Rs. 4,000 per year.

Model 2 : Dynamic Certainty (or Deterministic) Model allowing Shortages.

Model 2 is really an extension of Model 1 allowing shortages. In this model we owever, consider the planning horizon to be of infinite length and it is divided into eorder intervals each of length t. The costs involved are as follows :

$C_S =$ ordering cost per order

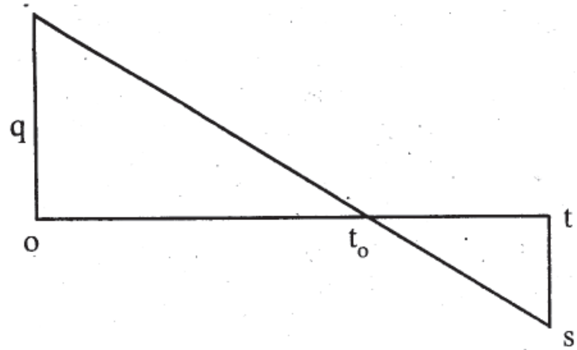
$C_1 =$ carrying cost per unit per unit time

$C_2 =$ shortage cost per unit short per unit time.

Demand is assumed to occur at the uniform rate of R units per unit time and the ead time is zero.

During each reorder interval a shortage of s units is allowed and it is backlogged. he policy is to place an order for $(q + s)$ units at the beginning of each reorder interval n order to bring the stock height to q.

As the planning horizon is of infinite length, we analyze the model over a reorder nterval. The diagrammatic representation of the model over the reorder interval (o, t) is s follows :



t_0 is the time point at which the stock on hand is zero.

The decision variables here are $q, s, t, t_o,$ or q, s, t_o, t_1 where $t_1 = t - t_o$.

Now, $q = Rt_o$

and $s = R(t - to) = Rt_1$.

Hence, there are two independent decision variables. Let us take these to be t_o and t_1. The total cost over (o, t) is given by

$$C(t_0, t_1) = \text{ordering cost} + \text{carrying cost} + \text{shortage cost}$$

$$= C_S + C_1 t_o \times \text{average inventory over } (o, t_o)$$

$$+ C_2(t - t_o) \times \text{average shortage over } (t_o, t - t_o).$$

$$= C_S + C_1 t_o \frac{q}{2} + C_2(t - t_o)\frac{s}{2}$$

$$= C_S + C_1 \frac{Rt_0^2}{2} + C_2 \frac{R^2 t_1^2}{2}$$

Since $t = t_o + t_1$ is itself unknown, we consider the total cost per unit time. This is given by

$$C^*(t_o, t_1) = \frac{C(t_0, t_1)}{t_0 + t_1}$$

The optimal values of t_o which minimize $C^*(t_o, t_1)$ satisfy

$$\frac{\partial C^*(t_0, t_1)}{\partial t_0} = 0 \text{ and } \frac{\partial C^*(t_0, t_1)}{\partial t_1} = 0,$$

which give

$$C_1Rt_o = C^*(t_o, t_1) \dots (1)$$

$$C_2Rt_1 = C^*(t_o, t_1) \dots (2)$$

Hence, $\dfrac{t_0}{t_1} = \dfrac{C_2}{C_1}$

or, $t_o = \dfrac{C_2 t_1}{C_1}$

Substituating this in (2) we get

$$\frac{C_2}{C_1}(C_1 + C_2)Rt_1{}^2 = C_s \frac{RC_2^2 t_1^2}{2C_1} + \frac{RC_2 t_1^2}{2}$$

$$= C_s + \frac{RC_2 t_1^2}{2C_1}(C_1 + C_2)$$

which gives optimal t, as

$$t_{10} = \sqrt{\frac{2C_1 C_s}{RC_2(C_1 + C_2)}}$$

Hence optimal values of t_o, q and s are

$$t_{00} = \frac{C_2}{C_1} t_{10} = \sqrt{\frac{2C_2 C_s}{RC_1(C_1 + C_2)}}$$

$$r_0 = Rt_{00} = \sqrt{\frac{2RC_2 C_s}{C_1(C_1 + C_2)}}$$

$$s_0 = Rt_{10} = \sqrt{\frac{2RC_1 C_s}{C_2(C_1 + C_2)}}$$

and minimum cost is

$$C^*(t_{00}, t_{10}) = \sqrt{\frac{2RC_s C_1 C_2}{C_1 + C_2}}.$$

<u>Particular case.</u>

If shortage cost C_2 is very large compared to carrying cost C_1, i.e., $\dfrac{C_1}{C_2} \approx 0$, then

$$t_{00} = \sqrt{\frac{2C_s}{RC_1\left(1+\frac{C_1}{C_2}\right)+1}} \approx \sqrt{\frac{2C_s}{RC_1}}$$

$$q_0 \approx \sqrt{\frac{2RC_s}{C_1}}$$

$$s_0 \approx 0$$

and minimum cost $= \sqrt{\frac{2RC_sC_1}{\frac{C_1}{C_2}+1}} \approx \sqrt{2RC_sC_1}$

Thus the model reduces to Model 1 with demand rate $R = D/T$.

Example : The demand of an item is uniform at a rate of 25 units per month. The fixed cost is Rs. 15 each time a production run is made. The production cost is Re. 1 per item, and the inventory carrying cost is Re. 0.30 per item per month. If the shortage cost is Rs. 1.50 per item per month, determine how often to make a production run and of what size is should be?

Solution : Here production rate is infinite so that the model is similar to Model 2.

The costs are

C_1 = Re. 0.30 per item per month

C_2 = Rs. 1.50 per item short per month

C_S = Rs. 15,00 per production run

b = production cost per item

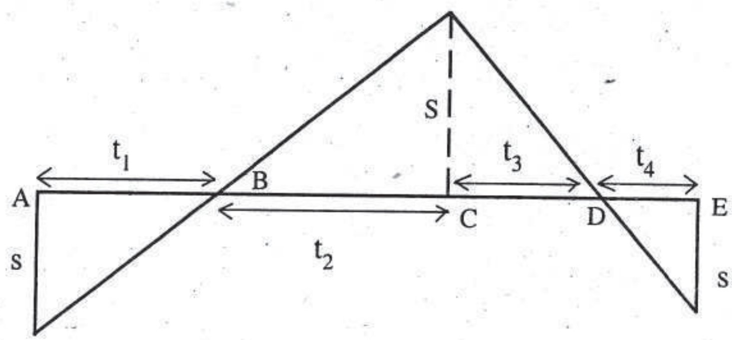
= Re. 1.00 per item

R = 25 units per month.

Total cost per unit length of production run is, therefore,

$C^*(t_o, t_1) + bR$, since Rt is the amount produced over (o, t), which defines a production run. Here t, t_o, t_1 have the same meaning as in Model 2. So,

Optimal length of a production run (i.e., of a reorder interval) $= t_{00} + t_{10}$



Production starts at A and stops at C. It again starts at E. At B and D stock on hand is zero. In the interval AC inventory piles up at the rate of $(K-R)$ items per unit time while in the region CE stock diminishes at the rate R items per unit time. As such, the relations among the decision variables s, S, t_1, t_2, t_3, t_4 are

$$s = (K-R)t_1 = Rt_4$$

$$S = (K-R)t_2 = Rt_3.$$

Hence out of 6 decision variables only two can be independently chosen. Let these be t_1 and t_2. The length of a cycle, therefore, is

$$t_1 + t_2 + t_3 + t_4 = \frac{K}{R}(t_1 + t_2).$$

Since this itself is to be decided upon, in order to find t_1 and t_2 we consider the cost per unit length of the production-inventory cycle.

The cost over a cycle is set-up cost + total carrying cost + total shortage cost

Now, total carrying cost over a cycle = $C_1 t_2 \times$ average inventory in BC

+ $C_1 t_3 \times$ average inventory in CD.

$$= C_1(t_2 + t_3)\frac{S}{2}$$

$$= \frac{C_1(K-R)Kt_2^2}{2R}$$

Similarly, total shortage cost over the cycle = $C_2 t_1 \times$ average shortage in AB

+ $C_2 t_4 \times$ average shortage in DE

$$= C_2(t_1 + t_4)\frac{S}{2}$$

$$= \sqrt{\frac{2C_2C_s}{RC_1(C_1+C_2)}} + \sqrt{\frac{2C_1C_s}{RC_2(C_1+C_2)}}$$

$$= \sqrt{\frac{2(C_1+C_2)C_s}{RC_1C_2}}$$

= 2.16 months.

And,

Optimal production quantity (i.e., order quantity) = $q_0 + s_0$

$$= \sqrt{\frac{2R(C_1+C_2)C_s}{C_1C_2}} = 54 \text{ items.}$$

Model 3 : Production inventory model.

Consider a production process which produces items at a finite rate of K items per unit time. The demand for the item occurs at an uniform rate of R items per unit time, $R < K$. The costs involved are as follows—

C_S = set up cost per production set up

C_1 = carrying cost per item per unit time

C_2 = shortage cost per item per unit time.

The production inventory policy is as follows :

Production is started as soon as there is a maximum shortage of s units. As a result backlog diminishes and becomes zero in t_1 units of time. Stock now starts to build up and reaches a maximum height S after t_2 units of time. At this point production is stopped and demand is met from the accumulated stock. After t_3 units of time the stock is completely exhausted and shortage starts piling up. When shortage or backlog reaches a maximum level s, say after t_4 units of time, production is again started. The process is repeated over and over again.

The iterval between two consecutive starting points of production is called a production-inventory cycle. Diagrammatically, the inventory situation over a cycle will be as follows :

$$= \frac{C_2(K-R)Kt_1^2}{2R}$$

And the set-up cost is C_S.

Hence, the cost per unit length of a cycle is

$$C(t_1, t_2) = \frac{C_s + \frac{1}{2}C_1\frac{(K-R)}{R}t_2^2 + \frac{1}{2}C_2\frac{(K-R)K}{R}t_1^2}{\frac{K}{R}(t_1, t_2)}$$

$$= \frac{RC_s + \frac{1}{2}(K-R)Kt_2^2 + \frac{1}{2}C_2(K-R)Kt_1^2}{R(t_1 + t_2)}$$

The optimal values of t_1 and t_2 which minimize $C(t_1, t_2)$ must satisfy

$$\frac{\partial C(t_1,t_2)}{\partial t_1} = 0 \text{ and } \frac{\partial C(t_1,t_2)}{\partial t_2} = 0,$$

which give

$$C_2(K-R)t_1 = C(t_1, t_2) \qquad \text{... (1)}$$

$$C_1(K-R)t_2 = C(t_1, t_2) \qquad \text{... (2)}$$

$$\therefore \quad t_2 = \frac{C_2}{C_1}t_1.$$

Substituting this in (1) we get

$$\frac{C_2}{C_1}(K-R)K(C_1+C_2)t_1^{\,2}$$

$$= RC_s + \frac{1}{2}(K-R)KC_1\frac{C_2^2}{C_1^2}\, t_1^{\,2} + (K-R)KC_2t_1^{\,2}$$

or, $$\frac{C_2}{C_1}(C_1+C_2)(K-R)t_1^{\,2}$$

$$= RC_s + \frac{1}{2}(K-R)KC_1\frac{C_2^2}{C_1^2}t_1^{\,2} + (K-R)KC_2t_1^{\,2}$$

This gives optimal value of t_1 as

$$t^*_1 = \sqrt{\frac{2C_sC_1R}{(C_1+C_2)K(K-R)}}$$

and hence optimal value of t_2 is

$$t^*_2 = \sqrt{\frac{2C_sC_2R}{(C_1+C_2)KR}}$$

The optimal values of the other decision variables are, therefore,

$$t^*_3 = \sqrt{\frac{2C_sC_2(K-R)}{(C_1+C_2)KR}}$$

$$t^*_4 = \sqrt{\frac{2C_sC_1(K-R)}{(C_1+C_2)KR}}$$

$$S^* = \sqrt{\frac{2C_sC_2R(K-R)}{(C_1+C_2)K}}$$

$$s^* = \sqrt{\frac{2C_sC_1R(K-R)}{(C_1+C_2)K}}$$

and the minimum cost is given by

$$C^* = \sqrt{\frac{2RC_sC_1C_2R(K-R)}{K(C_1+C_2)}}$$

Special cases

(i) If in the model we take production to be instantaneous, i.e., $k \to \infty$, then noting that $\frac{R}{K} \to \infty$ we get

$$t^*_1 \approx 0,\ t^*_2 \approx 0$$

$$t^*_3 = \sqrt{\frac{2C_sC_2(1-R/K)}{C_1(C_1+C_2)R}} \approx \frac{2C_sC_2}{C_1(C_1+C_2)R}$$

$$t^*_4 = \sqrt{\frac{2C_sC_1(1-R/K)}{C_2(C_1+C_2)R}} \approx \frac{2C_sC_1}{C_2(C_1+C_2)R}$$

$$S^* = Rt^*_3 \approx \sqrt{\frac{2C_s C_1 R}{C_2(C_1 + C_2)}}$$

$$s^* = Rt^*_4 \approx \sqrt{\frac{2C_s C_2 R}{C_1(C_1 + C_2)}}$$

and minimum cost $\approx \sqrt{\frac{2RC_s C_1 C_2}{2(C_1 + C_2)}}$

This gives Model 2.

(ii) In addition to $K \to \infty$ if we take C_2 to be very large compared to C_1 so that $\frac{C_1}{C_2} \approx 0$, then the model reduces to Model 1. This follows from the fact that Model 2 reduces to Model 1 for $\frac{C_1}{C_2} \approx 0$.

Example : The demand for an item in a company is 18,000 units per year, and the company can produce the item at a rate of 3,000 per month. The cost of one set-up is Rs. 500.00 and the holding cost of one uint per month is 15 paise. The shortage cost of one unit is Rs. 20.00 per year. Determine the optimum manufacturing quantity and the number of shortages. Also, determine the manufacturing time and the time between set-ups.

Solution : Here C_S = Rs. 500, C_1 = Re. 0.15, C_2 = Rs. 20, K = 3,000 units / month,

R = 18,000/12 units/month = 1500 units/month.

$\therefore$ Optimum quantity manufactured = $K(t^*_1 + t^*_2)$

$$= \sqrt{\frac{2C_s(C_1 + C_2)RK}{C_1 C_2(K - R)}}$$

= 4,670 units.

No. of shortages is

$s^* = (K - R)t^*_1$ = 193 units.

Manufacturing time = $t^*_1 + t^*_2$ = 0.13 year.

Time between set-ups— $t^*_1 + t^*_2 + t^*_3 + t^*_4$ = 0.26 year.

Model 4 : Multi-item deterministic model with constraints.

So far we have considered inventory models for a single item. However, a more realistic situation would be where the inventory manager stocks more than one type of item. Here we shall consider deterministic models for multi-item inventory.

Consider the following assumptions :

(i) n items with instantaneous production and no lead time.

(ii) R_i is the constant uniform demand rate for the i-th item ($i = 1, 2, ..., n$).

(iii) $C_2^{(i)}$ is the holding (or carrying) cost per unit quantity of the i-th item.

(iv) Shortages are not allowed (i.e., $C_2^{(i)} = 0$).

(v) $C_3^{(i)}$ is the set-up cost per production run for the i-th item.

(vi) q_i is the total quantity of the i-th item produced at the beginning of the production run.

We get the cost per unit time for the i-th item as :

$$C_i^*(t) = 1/2C_i^{(i)} R_i t + C_3^{(i)}/t \text{ or } C_i(q_i) = 1/2C_i^{(i)} q_i t + C_3^{(i)} R_i/q_i$$

Hence summing up these costs for $i = 1, 2, ..., n$, we get

$$C = \sum_{i=1}^{n} [1/2C_1^{(i)} q_i + C_3^{(i)} R_i/q_i] \text{ (cost equation)} \qquad ... (1)$$

To determine the optimum value of q_i ($i = 1, 2, ..., n$) so that the total cost C is minimum, we have the necessary conditions $C/q_i = 0$, $i = 1, 2, ..., n$. Therefore, we get

$$\frac{\partial C}{\partial q_i} = 1/2C_1^{(i)} - C_3^{(i)} R_i/q_i^2 = 0$$

which gives $q_i = \sqrt{2C_1^{(i)} R_i / C_1^{(i)}}$ ... (2)

Since $\partial^2 C/\partial q_1^{\,2} > 0$ for all q_i, the total cost C is minimum for q_i's given by (2). Hence the optimum value of q_i is given by

$$q_i = \sqrt{(2C_3^{(i)} R_i / C_1^{(i)}} \quad i = 1(1)n.$$

We now proceed to consider the effect of limitations, viz. (i) limitation on investment, (ii) limitation on stocked units, and (iii) limitation on warehouse floor space.

Model 4(a) : Limitation on investment

In this case, there is an upper limit, say M on the money to be invested on inventory.

Let $C_4^{(i)}$ be the unit price of i-th item. Then

$$\sum_{i=1}^{n} C_4^{(i)} q_i < M. \qquad \text{... (3)}$$

Now our problem is to minimize the total cost C given by the equation (1) subject to the additional constraint above. In this situation, two cases may arise :

Case 1. When $\sum_{i=1}^{n} C_4^{(i)} q^*_i \leq M$, q^*_i given by (2).

In this case, q^*_i given by (2) is the required optimal value of q_i, $i = 1(1)n$.

Case 2. When $\sum_{i=1}^{n} C_4^{(i)} q^*_i > M$, q^*_i given by (2).

In this case q^*_i ($i = 1, 2, ..., n$) given by (2) are not the required optimal values. Therefore, we shall use the Lagrange's multiplier technique as follows :

The Lagrangian function is

$$L = \sum_{i=1}^{n}\left(\frac{1}{2}C_1^{(i)} q_i + \frac{C_3^{(i)} R_i}{q_i}\right) + \left(\sum_{i=1}^{n} C_4^{(i)} q_i - M\right)$$

Here λ is the Lagrange multiplier.

The necessary condition for L to be minimum is $\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial \lambda} = 0$ ($i = 1, 2, ..., n$).

$$\text{Therefore, } \frac{\partial L}{\partial q_i} = \frac{1}{2}C_1^{(i)} - \frac{C_3^{(i)} R_i}{q_1^2} + \lambda C_4^{(i)} = 0, \ (i = 1, 2, ..., n). \qquad \text{... (4)}$$

$$\text{and } \frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} C_4^{(i)} q_i - M = 0 \qquad \text{... (5)}$$

$$\text{These equations give } q^*_1 = \sqrt{\left(\frac{2C_3^{(i)} R_i}{C_1^{(i)} + 2\lambda * C_4^{(i)}}\right)} \qquad \text{... (6)}$$

$$\text{and } \sum_{i=1}^{n} C_4^{(i)} q^*_i = M \qquad \text{... (7)}$$

The second equation implies that q^*_i must satisfy the investment constraint in equality sense.

Since q^*_i depends on λ^* (the optimal value of λ), λ^* can be found by systematic trial and error. By trying successive positive value of λ, the value of λ^* should result in simultaneous value of q^*_i satisfying (7). Thus determination of λ^* will automatically determine q^*_i.

The following interesting example will make the procedure clear :

Example : Consider a shop which produces three items. The items are produced in lots. The demand rate for each item is constant and can be assumed to be deterministic. No back orders are to be allowed.

The pertinent data for the items is given in the following table :

Item	1	2	3
Holding cost (Rs.)	20	20	20
Set-up cost (Rs.)	50	40	60
Cost per unit (Rs.)	6	7	5
Yearly demand rate	10,000	12,000	7,500

Determine approximately the Economic Order Quantities when the total value of average inventory levels of three items is Rs. 1000.

Solution : First of all we compute the optimal values q^*_i without considering the effect of restrication by using the formula (2). Thus, we get

$$q_i^* = \sqrt{\left(\frac{2\times 50\times 1000}{20}\right)},\ q_2^* = \sqrt{\left(\frac{2\times 40\times 12000}{20}\right)},\ q_3^* = \sqrt{\left(\frac{2\times 60\times 7500}{20}\right)}$$

$$= 100\sqrt{5} = 223 \text{ approx.} = 40\sqrt{30} = 216 \text{ approx.} = 150\sqrt{2} = 210 \text{ approx.}$$

Since the average optimal inventory at any time is $1/2\, q^*_i$, the investment on average inventory is obtained as

$$\sum_{i=1}^{n} C_4^{(i)}\,(1/2q_i^*) = \text{Rs. } (6\times 223/2 + 7\times 216/2 + 5\times 210/2) = \text{Rs. } 1950.00.$$

We observe that the amount Rs. 1950 is greater than the upper limit of Rs. 1000. Therefore, we try to find the suitable value of λ by trial and error method for computing q^*_i by using the formula (6).

If we set $\lambda = 4$ in (6) we get

$$q_i^* = \sqrt{\left(\frac{2\times 50\times 10000}{20+2\times 4\times 6}\right)} = 121,\ q_2^* = \sqrt{\left(\frac{2\times 40\times 12000}{20+2\times 4\times 7}\right)} = 112,$$

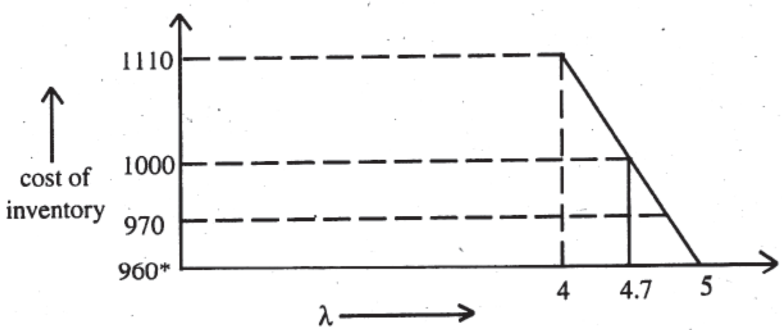
$$q_3^* = \sqrt{\left(\frac{2\times 60\times 7500}{20+2\times 4\times 5}\right)} = 113$$

And hence the cost of average inventory $= 6 \times \frac{121}{7} + 7 \times \frac{112}{2} + 5 \times \frac{123}{2} =$ Rs. 1112.50.

Again if we set $\lambda = 5$ in (6) we obtain $q^*_1 = 111$. $q^*_2 = 102$, $q^*_3 = 113$, and the corresponding cost of average inventory is Rs. 972.50 which is less than Rs. 1000.

From this we conclude that the most suitable value of λ lies between 4 and 5.

To find the most suitable value of λ, we draw a graph between cost of average inventory and the value of λ as shown in the following figure :



This graph indicates that $\lambda = 4.7$ (approx.) is the most suitable value corresponding to which the cost of inventory is approximately Rs. 999.50, which is sufficiently close to Rs. 1000.

For $\lambda = 4.7$, we obtain the required optimal values of three items as :

$$q^*_1 = 114.\ q^*_2 = 105,\ q^*_3 = 116.$$

Model 4(b) : Limitation on inventories

In this case, the upper limit of average number of units in stock is specified as N (say),, i.e., the average number of units in inventory should not exceed N. Since the average number of units at any time is $1/2\, q_i$, for $i = 1(1)n$, we have to minimize the cost C given by (1) subject to the condition

$$\frac{1}{2}\sum_{i=1}^{n} q_i < N. \quad \text{... (8)}$$

Now two possibilities may arise :

Case 1. If $\frac{1}{2}\sum_{i=1}^{n} q^*_i < N$, the optimum value of q_i's are $q^*_i (i = 1, 2, ..., n)$ given by formula (2)

Case 2. If $\frac{1}{2}\sum_{i=1}^{n} q_i > N$, then the optimal values given by (2) are no longer optimal. So we use Largrange's multiplier technique.

The Lagrangian function is given by

$$L = \sum_{i=1}^{n}\left(\frac{1}{2}C_1^{(i)} q_i + \frac{C_3^{(i)} R_i}{q_i}\right) + \lambda\left(\frac{1}{2}\sum_{i=1}^{n} q_i - N\right)$$

where $\lambda > 0$ is the Lagrange multiplier.

The optimum values of q_i are obtained by setting

$$\frac{\partial L}{\partial q_i} = \frac{1}{2}C_1^{(i)} - \frac{C_3^{(i)} R_i}{q_i^2} + \frac{\lambda}{2} = 0, \text{ and } \frac{\partial L}{\partial \lambda} = \frac{1}{2}\sum_{i=1}^{n} q_i - N = 0, \text{ for } i = 1, 2, ..., n.$$

Solving these two equations, we get

$$q^*_i = \sqrt{\left(\frac{2C_3^{(i)} R_i}{C_1^{(i)} + \lambda^*}\right)}, \quad i = 1, 2, ..., n \quad \text{... (9)}$$

$$\text{and} \quad \sum_{i=1}^{n} q^*_i = 2N. \quad \text{... (10)}$$

To obtain the values of q^*_1 from (9) we find the optimal value λ^* of λ by successive trial and error method, subject to the condition given by (10).

The following example will illustrate the procedure.

Example : A company producing three items has a limited storage space of averagely 750 items of all types. Determine the optimal production quantities for each item separately when the following information is given :

Product	1	2	3
Holding cost (Rs.)	0.05	0.02	0.04
Set-up cost (Rs.)	50	40	60
Demand rate (per unit)	100	120	75

Solution : Neglecting the restriction of the total value for inventory level, we find

$$q^*_1 = \sqrt{\left(\frac{2\times 50\times 100}{0.05}\right)},\ q_2^* = \sqrt{\left(\frac{2\times 40\times 120}{0.04}\right)},\ q_3^* = \sqrt{\left(\frac{2\times 60\times 75}{0.04}\right)}$$

$= 100\sqrt{20} = 447$ (nearly) $= 100\sqrt{48} = 6893$ (nearly) $= 100\sqrt{21.5} = 464$ (nearly). Therefore, the total average inventory is 1/2 (447 + 693 + 464) = 802 units.

But, we are given the storage capacity of 750 items per year and, therefore, we have to find the value of parameter λ by trial and error substitution process.

For $\lambda = 0.002$, we get $q^*_1 = 428$. $q^*_2 = 628$, $q^*_3 = 444$.

$\therefore$ Total average inventory becomes

$^1/_2$ [428 + 628 + 444] = 750 units,

which is equivalent to the given amount of average inventory 1.

Hence, the optimal production quantities for the three items are :

$q^*_1 = 428$ units, $q^*_2 = 628$ units, and $q^*_3 = 444$ units.

4(c) : Limitations on floor space (storage space)

In this model, the inventory system includes $n(> 1)$ items which are competing for a limited storage space. An interaction between the different items occurring due to this limitation can be included as an additional constraint.

Let A = the maximum storage area available for the n items,

a_j = storage area required per unit of the i-th item, and

q_i = the amount ordered of the i-th item.

Thus, the storage requirement constraint becomes :

$$\sum_{i=1}^{n} a_i q_i < A,\ q_i > 0,\ i = 1(1)n \qquad \text{... (11)}$$

The relevant inventory costs for each item are the same as in the case of Model 1. Thus, our problem becomes :

Minimize, $C = \sum_{i=1}^{n}\left(\frac{1}{2}C_1^{(i)}q_i + \frac{C_3^{(i)}R_i}{q_i}\right)$, subject to the constraint (11).

The Lagrange multipliers method yields the general solution of this problem. However, before applying this method, it is necessary to check whether the unconstrained value of q_i given by (2) satisfy the storage constraint. If not, the new optimal values of q_i must be determined which will satisfy the storage constraint in equality sense. This is done by first formulating the Lagrangian function :

$$L = \sum_{i=1}^{n}\left(\frac{1}{2}C_1^{(i)}q_i + \frac{C_3^{(i)}R_i}{q_i}\right) + \lambda\left(\sum_{i=1}^{n} a_i q_i - A\right)$$

where $\lambda > 0$ is the Lagrange multiplier.

Proceeding as in Model 4(a), we obtain the optimal values

$$q^*_i = \sqrt{\left(\frac{2C_3^{(i)}R_i}{C_1^{(i)} + 2\lambda^* a_i}\right)}, \quad i = 1, 2, ..., n \qquad ... (12)$$

and $$\sum_{i=1}^{n} a_i q^*_i = A. \qquad ... (13)$$

The second equation implies that q^*_i must satisfy the storage constraint in equality sense. The determination of λ^* by usual trial and error method automatically yields the optimum values q^*_i.

We illustrate this model by an example.

Example : Consider an inventory problem with three items. The parameters of the problem are given in the table below.

Item (i)	T_i (units)	$C_3^{(i)}$ (Rs.)	$C_1^{(i)}$ (Rs.)	$a_i(mt^2)$
1	20	100	30	1
2	40	50	10	1
3	30	150	20	1

Assume that the total available storage area is given by A = 25 mt^2. Find the optimal order quantities.

Solution : Substituting the given values in (12) we construct the following table:

λ	q_1	q_2,	q_3	$\Sigma a_i q_i - A$
0	11.5	20.0	21.2	27.7
5	10.0	14.1	17.3	16.4
10	9.0	11.5	14.9	10.4
15	8.2	10.2	13.4	6.6
20	7.6	8.9	12.2	3.7
25	7.1	8.2	11.3	1.6
30	6.7	7.6	10.6	– 0.1

We observe that the storage constraint is satisfied for some value of λ between 25 and 30. From the table we find that the value of λ is approximately 30. Hence the optimal order quantities are approximately $q^*_1 = 6.7$, $q^*_{21} = 7.6$ and $q^*_3 = 11.6$.

Model 5 : *EOQ* model with quantity discounts

When the inventory manager purchases an item he may be offered a discount on the purchase price per unit if he buys at least a specified amount of the item. This is known as quantity discount. The manager has to decide whether to avail the discount or not on the basis of the cost incurred or profit made.

Consider the *EOQ* model (Model 1). Let us introduce *P* as the purchase price per unit and let us redefine the carrying cost per unit per unit time as a fraction I of the value of one unit in inventory.

The value of one unit in inventory is defined as the total cost associated with procuring one unit. As the cost of procuring q (order quantity) units is $C_S + P_q$, the value of one unit in inventory $\dfrac{C_s + P_q}{q}$ or $\dfrac{C_s}{q} + P$.

Hence the total carrying cost over (0, *T*) is $\dfrac{qI}{2}\left(\dfrac{C_s + P_q}{q}\right)T = \dfrac{IC_sT}{2} + \dfrac{PI_q}{2}T$

Therefore, the total cost over (0, *T*) is

$C(q)$ = total ordering cost + total purchase cost + total carrying cost

$$= \frac{D}{q}C_s + PD + \frac{IC_sT}{2} + \frac{PI_qT}{2}$$

Setting $\frac{dC(q)}{q} = 0$ we obtain the optimal order quantity as

$$q^* = \sqrt{\frac{2DC_s}{PIT}}$$

and the minimum cost is

$$C(q^*) = \frac{C_s IT}{2} + PD + \sqrt{2DC_s PIT}$$

Model 5(a) : *EOQ* model with one price-break

Consider a purchasing situation where only one quantity discount is available Such a situation may be represented as follows :

Purchase price (P) per unit	Quantity range
P_1	$q < b$
P_2	$q \geq b$

where $P_1 > P_2$ and b is the quantity beyond which discount is available.

Then the total cost over $(0, T)$ for $q < b$ is

$$C_1(q) = \frac{D}{q}C_S + P_1 D + \frac{IC_s T}{2} + \frac{P_1 Iq}{2}T$$

and for $q \geq b$ is

$$C_2(q) = \frac{D}{q}C_S + P_2 D + \frac{IC_s T}{2} + \frac{P_2 I_q}{2}T.$$

$q^*_i = \sqrt{\frac{2DC_s}{P_i IT}}$ is the optimal order quantity minimizing $C_i(q)$, $i = 1, 2$. Since P_1 : P_2 clearly $q^*_1 < q^*_2$. The steps for finding the optimal order quantity are as follows :

Step 1. Compute q^*_2. If $q^*_2 > b$, then q^*_2 is the optimal order quantity. [This i because since $P_1 > P_2$, $C_1(q^*_1) = \sqrt{2DC_s P_1 IT} > \sqrt{2DC_s P_2 IT} = C_2(q_2^*)$].

If $q^*_2 < b$, go to step 2.

Step 2. Compute q^*_1, $C_1(q^*_1)$ and $C_2(b)$.

If $C_1(q^*_1) < C_2(b)$, then q^*_1 is the optimal order quantity, else $q = b$ is the optima order quantity. [This follows from the following argument—As $C_2(q)$ is a convex functio of q and $q^*_2 < b$, over the range $[b,)$ of q, $C_2(q)$ is a non-decreasing function of q so that

$$\min_{q \geq b} C_2(q) = C_2(b).$$

Hence for $C_1(q^*_1) < C_2(b)$, q^*_1 is optimal, and if $C_1(q^*_1) > C_2(b)$, $q = b$ is optimal.]

Example : Consider an item on which incremental quantity discounts are available. The first hundred units cost Rs. 100 each and additional units cost Rs. 95 each. For this item, demand = 500 units per year, inventory carrying charge = 20% of average inventory valuation per annum, and procurement cost = Rs. 50. Determine, the *EOQ*.

Solution : Given that $D = 500$ units per year, $C_3 =$ Rs. 50.00 and $I =$ Rs. 0.20 per year. Quantity discounts are given below :

Unit cost	Quantity
Rs. 100.00	$0 < q_1 < 100$
Rs. 95.00	$100 < q_2$

$$\text{Now } q^*_2 = \sqrt{\left(\frac{2C_3D}{P_2I}\right)} = \sqrt{\left(\frac{2 \times 50 \times 500}{95 \times 0.20}\right)} = 100\sqrt{\left(\frac{5}{19}\right)} = 51 \text{ units.}$$

Since $q^*_2 < b$ (= 100), we next compute q^*_1 to have

$$q^*_1 = \sqrt{\left(\frac{2C_3D}{P_1I}\right)} = \sqrt{\left(\frac{2 \times 50 \times 500}{100 \times 0.20}\right)} = 50 \text{ units.}$$

We now compute the costs :

$$C(q^*_1) = 50 \times \frac{500}{50} + 500 \times 100 + 100 \times 0.20 \times \frac{50}{2} = \text{Rs. } 51{,}000$$

$$C(b) = 50 \times \frac{500}{50} + 500 \times 95 + 95 \times 0.20 \times \frac{100}{2} = \text{Rs. } 48{,}700.$$

Since $C(b) < C(q^*)$, the optimum order quantity is $q^* = b = 100$ units.

Model 5(b) : *EOQ* model with two price breaks

Here we have

$$\begin{aligned} \text{purchase cost } (P) \text{ per unit } &= P_1 \text{ if } q < b_1 \\ &= P_2 \text{ if } b_1 \leq q < b_2 \\ &= P_3 \text{ if } q \geq b_2, \end{aligned}$$

where $P_1 > P_2 > P_3$ and $b_1 < b_2 < b_3$.

$C_i(q)$ denote the total cost over $(0, T)$ when $P = P_i$ and q^*_i be the corresponding optimal order quantity, $i = 1, 2, 3$.

Steps to find optimal order quantity

Step 1. Compute q^*_3 and compare with b_2.

(i) If $q^*_3 > b_2$, then the optimum purchase quantity is q^*_3, (ii) if $q^*_3 < b_2$, then go to step 2.

Step 2. Compare q^*_2. Since $q^*_3 < b_2$, q^*_2 is also less then b_2 (because $q^*_1 < q^*_2 < q^*_3 < \dots < q^*_n$, in general). Thus, thère are only two possibilities when $q^*_2 < b_2$, viz. either $q^*_2 > b_1$ or $q^*_2 < b_1$.

(i) If $q^*_2 < b_2$ but $> b_1$, then proceed as in the case of one price break only. That is, compare the costs $C_2(q^*_2)$ and $C_3(b_2)$ to obtain the optimum purchase quantity. The quantity with lower cost will naturally be the optimum.

(ii) If $q^*_2 <$ (b_2 and b_1 both), then go to step 3.

Step 3. If $q^*_2 <$ (b_2 and b_1 both), then find q^*_1 which will automatically satisfy the inequality $q^*_1 < b_1$. Compare the cost $C_1(q^*_1)$ with $C_2(b_1)$ and $C_3(b_2)$ both to determine the optimum purchase quantity.

Example : A shopkeeper has a uniform demand of an item at the rate of 50 items per month. He buys from supplier at a cost of Rs. 6 per item and the cost of ordering is Rs. 10 each time. If the stock-holding costs are 20% per year of stock value, how frequently should he replenish his stocks?

Now suppose the supplier offers at 5% discount on orders between 200 and 999 items and a 10% discount on orders exceeding or equal to 1000. Can the shopkeeper reduce his costs by taking advantage of either of these discounts?

Solution : We are given that : $R = 600$ items per year, $C_3 =$ Rs. 10 per order, $C_1 =$ Rs. $(6 \times 0.20) =$ Rs. 1.20.

$$\therefore \quad q^* = \sqrt{\left(\frac{2C_3D}{P_1I}\right)} = \sqrt{\left(\frac{2 \times 50 \times 500}{100 \times 0.20}\right)} = 100 \text{ items.}$$

From this we observe that the shopkeeper must replenish the inventory after every two months, because 100 items are sufficient to meet the demand of two months only.

The total annual cost includes the fixed cost, set-up cost and stock holding costs.

In this case, the fixed cost is Rs. 6 × 600 = Rs. 3,600. Also, since each time 100 items are ordered, there will be six orderings throughout the year and hence the replenishment cost is Rs. 60.00.

But, the average inventory throughout the year is 100/2 = 50 units.

∴ Average inventory carrying cost = 50 × 0.2 × 6 = Rs. 60.00.

Hence the total cost = Rs. 36.00 + Rs. 60 + Rs. 60 = Rs. 3720.00.

In the case of quantity discounts, we have the following formulation :

Quantity	Unit cost (Rs.)
$0 \le q_1 < 200$	6.00
$0 \le q_2 < 1000$	5.70 (5% discount)
$1000 \le q_3$	5.40 (10% discount)

Therefore, ∴ $$q^*_3 = \sqrt{\left(\frac{2C_3D}{P_3}\right)} = \sqrt{\left(\frac{2 \times 10 \times 600}{(5.40) \times (0.20)}\right)} = 110 \text{ units.}$$

Since $q^*_3 < b_2$ (= 1000), we next compute q^*_2.

$$\therefore \quad q^*_2 = \sqrt{\left(\frac{2C_3D}{P_2I}\right)} = \sqrt{\left(\frac{2 \times 10 \times 600}{(5.70) \times (0.20)}\right)} = 105 \text{ units.}$$

Again, since q^*_1 (= 200), we next compute q^*_1.

$$\therefore \quad q^*_1 = \sqrt{\left(\frac{2C_3D}{P_1I}\right)} = \sqrt{\left(\frac{2 \times 10 \times 600}{(6.00) \times (0.20)}\right)} = 100 \text{ units.}$$

Now compute,

$$C(q^*_1) = 10 \times \frac{600}{100} + 600 \times 6 + (0.20) \times 6 \times \frac{100}{2} = \text{Rs. } 3720.$$

$$C(b_1) = 10 \times \frac{600}{200} + 600 \times (5.70) + (0.20) \times (5.70) \times \frac{200}{2} = \text{Rs. } 3564.$$

$$C(b_2) = 10 \times \frac{600}{100} + 600 \times (5.40) + (0.20) \times)5.40) \times \frac{1000}{2} = \text{Rs. } 3786.$$

Since $C_2(b_1) < C_1(q^*_1) < C_3(b_2)$, the optimum purchase quantity is $q^* = b_1 = 200$ units.

Hence the shopkeeper should accept the offer of 5% discount only, because in doing so his net saving during the year would be = Rs. 3720 – Rs. 3564 = Rs. 156.

Model 5(c) : *EOQ* model with *n* price breaks

Here the purchase price per unit is given by

Range of quantity :	$0 \leq q_1 < b_1$	$b_1 \leq q_2 < b_2$	: $b_{n-1} \leq q_n$
Purchase price per unit P :	P_1	P_2	: P_n.

Let $C_i(q)$ be the total cost corresponding to $P = P_i$, with optimum purchase quantity q^*_i, $i = 1(1)n$. Then the following general decision rules apply :

1. First, compute q^*_n. If $< q^*_n \geq b_{n-1}$, then the optimum purchase quantity is q^*_n.

2. If $q^*_n < b_{n-1}$, then compute q^*_{n-1}. If q^*_{n-2}, $(b_{n-2} \leq q^*_{n-1} < b_{n-1})$, then proceed as in the case of one price break, i.e., the optimum purchase quantity is determined by comparing $C_{n-1}(q^*_{n-1})$ with $C_n(b_{n-1})$.

3. If $q^*_n < b_{n-2}$, then compute q^*_{n-2}. If $q^*_{n-2} \geq b_{n-3}$, then proceed as in the case of two price break, i.e., the optimum purchase quantity is determined by comparing $C_{n-2}(q^*_{n-2})$ with $C_{n-1}(b_{n-2})$ and $C_n(b_{n-1})$.

4. If $q^*_{n-2} < b_{n-3}$, then compute q^*_{n-3}. If $q^*_{n-3} \geq b_{n-4,}$ then compare $C_{n-3}(q^*_{n-3})$ with $C_{n-2}(b_{n-3})$, $C_{n-1}(b_{n-2})$ and $C_n(b_{n-1})$.

5. Continue is this way until $q^*_{n-i} \geq b_{n-(i+1),}$ $[0 \leq i \leq n-1]$, and then compare $C_{n-i}(q^*_{n-i})$ with

$$C_{n-i+1}(b_{n-i}), C_{n-i+2}(b_{b-i+1}), C_{n-i+3}(b_{n-i+2}), ..., C_n(b_{b-1}).$$

This procedure will involve a finite number of steps, in fact at the most n, where n denotes the number of price ranges.

Example : Determine an optimal ordering rule for the following case : $D = 6{,}000$ units in one month, $C_3 =$ Rs. 50 per order $I = 0.02$ per month, where

$P_1 =$ Rs. 1.25	for	$0 \leq q_1 < 100$
$P_2 =$ Rs. 1.20	for	$100 \leq q_2 < 300$
$P_3 =$ Rs. 1.00	for	$5000 \leq q_4 < 1{,}000$
$P_5 =$ Rs. 0.95	for	$1000 \leq q_5 < 2{,}000$
$P_6 =$ Rs. 0.90	for	$q^6 \geq 2{,}000$.

Solution : Since

$$q^*_6 = \sqrt{\left(\frac{2C_3D}{IP_6}\right)} = \sqrt{\left(\frac{2\times 50\times 6000}{(0.20)\times(0.90)}\right)} = \frac{10{,}000}{\sqrt{3}} = 5780 \text{ units (approx.),}$$

which is greater than b_5 (= 2,000), the optimal order quantity is $q^*_6 = 5780$ units.

Model 6 : Dynamic demand inventory model

So far we have discussed inventory models in which the demand is uniform. But, in actual practice, we come across non-uniform demand such as having rising or falling trend and/or depicting the seasonal influences.

According to M. Kerner, the following procedure is adopted for scheduling of known but irregular batchwise demand.

Procedure : For each month n (starting with $n = 1$), the condition $n^2R_{n+1} < (C_3/IP)$ is checked, where R_{n+1} is the requirement for the next month, C_3 is the setup cost, and IP is the inventory carrying cost.

So long as the condition is satisfied (i.e., answer is 'Yes') n is increased by 1 to take the next month into consideration. But, as soon as the above condition is not satisfied (i.e., answer is 'No'), there is an end of the particular grouping, and the following month is taken as a new month 1 to proceed further in a like manner.

The following example will make the procedure clear.

Example : The monthly requirement schedule for a product is given below :

Month	1	2	3	4	5	6	7	8	9	10	11	12
Requirement	100	150	10	70	90	180	2	98	100	200	140	160

Unit price (P) = Rs. 15, set-up cost (C_3) = Rs. 150, and inventory carrying cost is 30% of the annual average inventory value.

Determine an optimum plan of setups and batch sizes.

Solution : We are given that C_3 = Rs. 150, C = Rs. 15, and I = 30% of annual average inventory value = 0.32/12 = 0.025 monthly.

Hence $\frac{C_3}{IP} = \frac{150}{0.025 \times 15} = 400.$

The working procedure may be tabulated as follows :

Month	Requirement	n	n^2R^{n+1}	Is $n^2R^{n+1} < 400$?	Action
1	100	1	150	Yes	I set-up
2	150	2	40	No	I set-up
3	10	3	630	Yes	I set-up
Size of I set-up	Total = 260				Set-up again in month 4
4	70	1	90	Yes	II set-up
5	90	2	720	No	II set-up
Size of II set-up	Total = 160				Set-up again in month 6
6	180	1	2	Yes	III set-up
7	2	2	392	Yes	III set-up
8	98	3	900	No	III set-up
Size of III set-up	Total = 280				Set-up again in month 9
9	100	1	200	Yes	IV set-up
10	200	2	560	No	IV set-up
Size of IV set-up	Total = 300				Set-up again in month 11
11	140	1	160	Yes	V set-up
12	160	2	–	–	V set-up

There are five setups with respective sizes of 260, 160, 280, 300, 300. The last setup could possibly take up some of the next years requirements.

II. Probabilistic Inventory Models

A probabilistic or risk inventory model is a model in which demand is a random variable following some probabilistic law.

Model 7 : Static risk model with no lead time

This is a one-period model where only one order is placed during the period. A typical example of this model is the newsboy problem. The problem is as follows :

A newspaper vendor starts his day with Q newspapers in hand. The demand for the newspaper during the day is known to be random. So at the end of the day the vendor may find some excess newspapers in hand or may be facing a shortage.

Accordingly, he incurs excess cost or shortage cost. His problem is, therefore, to find Q so as to minimize his total expected cost.

Let Q = order quantity, order being placed at the beginning of the period

C_1 = excess cost per unit quantity of item in excess

C_2 = shortage cost per unit quantity of item short

D = demand during the period with cumulative distribution function $F(d)$, $d \geq 0$.

The cost incurred is

$$C_1(Q-d) \text{ if } d \leq Q$$

and $$C_2(d-Q) \text{ if } d > Q.$$

Let $C(Q)$ denote the total expected cost.

Case 1. D is a discrete random variable with p.m.f. $f(d)$, $d \geq 0$. In this case

$$C(Q) = C_1 \sum_{d=0}^{Q} (Q-d)f(d) + C_2 \sum_{d=Q+1}^{Q} (d-Q)f(d).$$

The optimal value of Q which minimizes $C(Q)$ must satisfy

$$C(Q) \leq C(Q+1) \qquad \text{... (1)}$$

and $$C(Q) \leq C(Q-1) \qquad \text{... (2)}$$

Now,

$$C(Q+1) - C(Q) = C_1 \left\{ \sum_{d=0}^{Q+1} (Q+1-d)f(d) - \sum_{d=0}^{Q} (Q-d)f(d) \right\} + C_2 \left\{ \sum_{d=Q+2}^{\infty} (d-Q-1)f(d) - \sum_{d=0}^{Q} (Q-d)f(d) \right\}$$

$$= C_1 \left\{ \sum_{d=0}^{Q+1} (Q+1-d)f(d) - \sum_{d=0}^{Q} (Q-d)f(d) \right\} + C_2 \left\{ \sum_{d=Q+1}^{\infty} (d-Q-1)f(d) - \sum_{d=Q+1}^{Q} (d-Q)f(d) \right\}$$

$$= C_1 F(Q) - C_2\{1 - F(Q)\}$$

$\therefore$ From (1) we have

$$(C_1 + C_2)F(Q) \geq C_2.$$

or, $\quad F(Q) \ge \dfrac{C_2}{C_1 + C_2}$ ... (3)

Similarly, (2) gives

$$F(Q-1) \le \frac{C_2}{C_1 + C_2} \quad \text{... (4)}$$

Thus, optimal Q must satisfy

$$F(Q-1) \ge \frac{C_2}{C_1 + C_2} F(Q),$$

i.e., optimal Q is the smallest Q satisfying $F(Q) \ge \dfrac{C_2}{C_1 + C_2}$.

Case 2. D is an absolutely continuous random variable with p.d.f. $f(d)$, $d \ge 0$. Here

$$C(Q) = C_1 \int_0^Q (Q-d) f(d) dd + C_2 \int_Q^\infty (d-Q) f(d) dd.$$

The optimal value of Q satisfies $\dfrac{d}{dQ} C(Q) = 0$, which gives, after some simplification,

$$F(Q) = \frac{C_2}{C_1 + C_2}$$

Example (Newspaper-Boy Problem) **:**

A newspaper-boy buys papers for Rs. 2.60 each and sells them for Rs. 3.60 each. He cannot return unsold newspapers. Daily demand has the following distribution :

No. of customers (d)	:	23	24	25	26	27	28	29	30	31	32
Probability $P(d)$	:	.01	.03	.06	.10	.20	.25	.15	.10	.05	.05

If each day's demand is independent of the previous day's demand, how many papers should he order each day?

Solution : Letting C_1 = excess cost/paper, C_2 = shortage cost/paper, the optimal order quantity Q_{opt} is the smallest Q satisfying

$$\sum_{d=23}^{Q} p(d) \ge \frac{C_2}{C_1 + C_2}$$

Since the planning period is of infinite length, we analyze the model considering the cost over a reorder interval.

As demand X is a random variable, we may have one of the following situations in a reorder interval as indicated in the figures below—

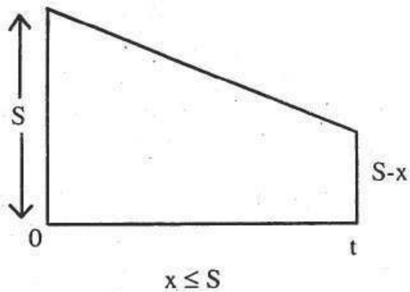


Fig. 1

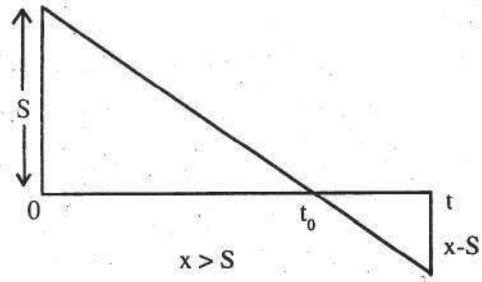


Fig. 2

We note that in figure 1 only carrying cost is incurred, while in figure 2 both carrying and shortage costs occur.

Assuming mean demand rate to be constant,

$$\text{average inventory} = \frac{2S+x}{2} \text{ for } x \leq S.$$

$$= \frac{S}{2} \text{ for } x > S$$

and $\quad$ average shortage $= 0$ for $x \leq S$

$$= \frac{x-S}{2} \text{ for } x > S.$$

Hence, the total cost over (o, t) is given by

$$C_1\left(S+\frac{x}{2}\right)t \text{ if } x \leq S$$

and $\quad C_1 \frac{S}{2}t_o + C_2\left(\frac{x-S}{2}\right)(t-t_o) \text{ if } x > S.$

i.e., $\quad C_1\left(S+\frac{x}{2}\right)t \text{ if } x \leq S$

and $\quad C_1 \frac{S^2}{2x}t + C_2 \frac{(x-S)^2}{2x}t \text{ if } x > S,$

But, C_1 = Rs. 2.60, and C_2 = Rs. 3.60—Rs. 2.60 = Rs. 1.00.

Hence $\sum_{d=23}^{Q} p(d) \geq \frac{1}{3.60}$ which gives $Q_{opt} = 27$.

Example : A baking company sells cake by the kg. weight. It makes a profit of Rs. 5.00 a kg. on each kg. sold on the day it is baked. It disposes of all cake not sold on the date it is baked at a loss of Rs. 1.20 a kg. If demand is known to be rectangular between 2000 and 3000 kg., determine the optimal daily amount baked.

Solution : We know that the optimal value of Q, the daily amount baked, is the smallest Q satisfying

$$\int_0^Q f(x)dx \geq \frac{C_2}{C_1 + C_2} \qquad \text{... (1)}$$

where

C_1 = excess cost/Kg. = Rs. 1.20, C_2 = shortage cost/Kg = Rs. 5.00,

$$f(x) = \frac{1}{1000}, \quad 2000 < x < 3000.$$

Substituting these values in (1) we get

$$\left[(C_1 + +\left\{\int_0^S\left(S + \frac{x}{2}\right)\sum_{x=S}^{\infty}\frac{C_2}{10 + C_2} = \sum_{x=S+1}^{\infty}\frac{p(x)}{x}dx = \frac{8}{10}\right\}\right]\begin{cases}0.1 & 0 \leq x \leq 10\\ 0 & \text{otherwise}\end{cases}$$

which gives $Q = 2807$ kg.

Model 8. Dynamic risk model with no lead time

Consider an inventory model with a planning period of infinite length, which is divided into reorder intervals of length t time units, t being known. Demand during each reorder interval is known to be a random variable X with c.d.f. $F(x)$, $x \geq 0$. It is assumed that replenishment is instantaneous on ordering.

The inventory policy is to place an order at the beginning of each reorder interval so as to bring the stock height instantaneously to S. Any shortage during a reorder interval is backlogged.

The inventory manager, therefore, has to decide upon the value of S.

Suppose the following costs are given :

C_1 = carrying cost per unit per unit time

C_2 = shortage cost per unit short per unit time since $\frac{S}{t_0} = \frac{x-S}{t-t_0}$, which gives

$$t_o = \frac{S}{x}\, t.$$

Case 1. X is a discrete random variable with p.m.f. $f(x)$, $x \geq 0$.

In this case,

$$C(S) = C_1 \sum_{x=0}^{S}\left(S+\frac{x}{2}\right)f(x) + C_1 \frac{S^2 t}{2} \sum_{x=S+1}^{\infty} \frac{f(x)}{x} + \frac{C_2 t}{2} \sum_{x=S+1}^{\infty} \frac{(x-S)^2}{x} f(x).$$

The optimal value of S, which minimizes $C(S)$, satisfies

$$C(S) \leq C(S+1) \qquad \text{... (1)}$$

and $$C(S) \leq C(S-1) \qquad \text{... (2)}$$

Now, $$C(S+1) - C(S) = t\left[(C_1 + C_2) + \left\{F(S) + \left(S+\frac{1}{2}\right)\sum_{x=S+1}^{\infty}\frac{f(x)}{x}\right\} - C_2\right],$$

so that (1) reduces to

$$F(S) + \left(S+\frac{1}{2}\right)\sum_{x=S+1}^{\infty}\frac{f(x)}{x} \geq \frac{C_2}{C_1 + C_2} \qquad \text{... (3).}$$

Similarly, from (2) we get

$$F(S-1) + \left(S-\frac{1}{2}\right)\sum_{x=S}^{\infty}\frac{f(x)}{x} \leq \frac{C_2}{C_1 + C_2} \qquad \text{... (4)}$$

Thus, optimal value of S must simultaneously satisfy (3) and (4).

Case 2. X is an absolutely continuous random variable with p.d.f. $f(x)$, $x \geq 0$. Here

$$C(S) = C_1 t \int_0^S \left(S+\frac{x}{2}\right)f(x)dx + C_1 \frac{tS^2}{2}\int_S^{\infty}\frac{1}{x} f(x)dx + \frac{C_2 t}{2}\int_S^{\infty}\frac{(x-S)^2}{x} f(x)dx.$$

$C(S)$ will, therefore, be minimum for a value of S satisfying $\frac{dC(S)}{dS} = 0$, which gives

$$F(S) + S\int_S^{\infty}\frac{f(x)}{x}dx = \frac{C_2}{C_1 + C_2} \qquad \text{... (5)}$$

Thus, optimal value of S satisfies (5).

Example : Let the probability density of demand of a certain item during a week be

$$f(x) = \begin{cases} 0.1 & 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

This demand is assumed to occur with a uniform pattern over the week. Let the unit carrying cost of the item in inventory be Rs. 2.00 per week and unit shortage cost be Rs. 8.00 per week. How will you determine the optimal order level of the inventory?

Solution : Since $f(x) = 0.1$, $0 \le x \le 10$, C_1 = Rs. 2.00, C_2 = Rs. 8.00, then if S denotes the maximum stock height, optimal S satisfies

$$\int_0^S (0.1)dx + S\int_S^{10} \frac{0.1}{x} dx = \frac{8}{10}$$

or, $0.1\,(S - S\log S + 2.3S) = 0.8$, or, $3.3S - S\log S - 8 = 0$.

The solution of this equation is obtained by trail and error method which gives $S = 4.5$.

Example : The probability distribution of monthly sales of a certain item is as follows :

Monthly sales	:	0	01	2	3	4	5	6
Probability	:	0.02	0.05	0.30	0.27	0.20	0.10	0.06

The cost of carrying inventory is Rs. 10.00 per unit per month. The current policy is to maintain a stock of four items at the begining of each month. Assuming that the cost of shortage is proportional to both time and quantity short, obtain the imputed cost of a shortage of one item for one time unit. (Because the problem is stated in discrete units, the answer will consist of a range of values for the imput cost).

Solution : (1) Optimum stock height $S = 4$ items,

(2) carrying cost C_1 = Rs. 10.00 per item per month,

(3) the probability $p(x)$ for sale x in each month is as follows :

$p(0)$	$p(1)$	$p(2)$	$p(3)$	$p(4)$	$p(5)$	$p(6)$
0.2	0.05	0.30	0.27	0.20	0.10	0.06,

(4) the shortage cost C_2 is to be determined,

(5) the range of monthly sales x is given from 0 to 6 times in discrete units (not from 0 to ∞ here).

We know that optimal S must satisfy

$$\sum_{x=0}^{S-1} p(x) + (S - 1/2) \sum_{x=S}^{\infty} \frac{p(x)}{x} \le \frac{C_2}{C_1 + C_2} \sum_{x=0}^{S} p(x) + (S + 1/2) \sum_{x=S+1}^{\infty} \frac{p(x)}{x}$$

Now least value of C_2 can be determined by letting

$$\frac{C_2}{10 + C_2} = \sum_{x=0}^{3} p(x) + (S - 1/2) \sum_{x=S}^{\infty} \frac{p(x)}{x}$$

Therefore, substituting the given values, we get

$$\frac{D}{Q} C_s + C_1 \frac{Q}{2} = \sum_{x=0}^{3} p(x) + (4 - 1/2) \sum_{x=4}^{6} \frac{p(x)}{x}$$

$$= [p(0) + p(1) + p(2) + p(3)] + \frac{7}{2}\left[\frac{p(4)}{4} + \frac{p(5)}{5} + \frac{p(6)}{6}\right]$$

$$= (0.02 + 0.05 + 0.03 + 0.27] + \frac{7}{2}\left[\frac{0.20}{4} + \frac{0.10}{5} + \frac{0.06}{6}\right] = 0.92.$$

$\therefore$ Least value of $C_2 = 9.2/.08 =$ Rs. 115.

Similarly, the greatest value of C_2 can be determined by letting

$$\frac{C_2}{C_1 + C_2} = \sum_{x=0}^{S} p(x) + (S + 1/2) \sum_{x=S+1}^{\infty} \frac{p(x)}{x}$$

Substituting the given values, we get

$$\frac{C_2}{10 + C_2} = \sum_{x=0}^{4} p(x) + (4 + 1/2) \sum_{x=5}^{6} \frac{p(x)}{x}$$

$$= 0.84 + 9/2 \times 0.03 = 0.975.$$

$\therefore$ Greatest value of $C_2 = 9.75/0.25 =$ Rs. 390.

Hence, the required range of values for the imputed cost C_2 is

Rs. $115 \le C_2 \le$ Rs. 390.

Model 9. Dynamic risk inventory models with lead time

Dynamic risk models with lead time may be classified into two broad classes viz. (i) fixed-order quantity system or Q system, and (ii) fixed reorder period system or P system.

Model 9(a) : Q-system : The Q system is a continuous review model. Here the policy is to place an order for Q units whenever the stock on hand reaches a minimum level r, say, called the reorder level. The ordered quantity isreceived at the end of the lead time period (L).

Since demand is probabilistic in nature, there exists the possibility of facing a shortage during the lead time. In order to reduce the chance of shortage, a safety stock w, also known as buffer stock or reserve stock, is maintained. The safety stock w is the the difference between the reorder level r and the expected lead time demand (d_L), i.e.,

$$w = r - d_L, \text{ or } r = d_L + w.$$

(If L be a random variable, d_L is the expected demand over average lead time)

The model is also referred to as the (S, s) system, where S denotes the maximum stock height and s the reorder level.

Let (O, T) be the planning period and D be the expected demand over (O, T). Let Y be the demand over the lead time (or average lead time). Let Y be a random variable with c.d.f. $F(y)$, $y \geq 0$.

Let C_s = ordering cost per order

C_1 = carrying cost per unit quantity per unit time

C_2 = shortage cost per unit short per unit time.

Then,

$$\text{total order cost over } (O, T) = \frac{D}{Q} C_S.$$

The total carrying cost for normal stock is $C_1 \frac{Q}{2} T$. However, we face an additional carrying cost for maintaining a safety stock against possible shortage, and this is given by $C_1(r - d_L)T = C_1 wT$. Hence, total carrying cost over (O, T) is

$$C_1 \frac{Q}{2} T + C_1 wT.$$

As shortage occurs during a lead time period if $r < Y$, i.e., $d_L + w < Y$, and there are in all $\frac{D}{Q}$ lead time periods in the planning horizon, the total shortage cost will be

$$Cs \frac{D}{Q_d} \sum_{L+w}^{\infty} (y - d_L - w) f(y) dy.$$

Hence, the total cost over (O, T) is .

$$\frac{D}{Q} C_S + C_1 \frac{Q}{2} T + C_1 wT + C_3 \frac{D}{Q_d} \sum_{L+w}^{\infty} (y - d_L - w) dF(y) dy.$$

The decision variables are Q and w, which are assumed to be independent. A good approximation of the order quantity Q is given by the *EOQ* formula of Model 1. Using this value of q, the optimal value of w which minimizes the total cost can be obtained.

Example : A certain item has an annual demand of 2000 units. The cost of placing an order is Rs. 400 and the annual carrying cost is Rs. 10 per unit. The costs of stockout are estimated to average Rs. 10. The demand during lead time tends to be randomly distributed throughout the year, so that a Poisson distribution may be assumed. The are 250 working days per year and lead time is 5 working days.

Demand during lead time (d) :	70	75	80	85	90	95	100
Probability $p(d)$:	0.02	0.14	0.23	0.24	0.21	0.120	.04

Determine the optimal order quantity and reorder level.

Solution : We are given that

D = 2000 units/year, C_3 = Rs. 400/order

C_1 = Rs. 10 per unit/per year, C_2 = Rs. 10 per unit, and lead time $(L) = 5$ days. The optimal value of order quantity cand be obtained as :

$$Q^* \quad \sqrt{\left(\frac{2DC_3}{C_1}\right)} = \sqrt{\frac{2 \times 2000 \times 400}{10}} = 400 \text{ units.s}$$

The expected number of units demanded (denoted by d_L) during the lead time is given by

$$d_L = d_1 p(d_1) + d_2 p(d_2) + d_3 p(d_3) + d_4 p(d_4) + d_5 p(d_5) + d_6 p(d_6 + d_7 p(d_7)$$

$= (70 \times .02) + (75 \times .14) + (80 \times .23) + (85 \times .24) + (90 \times .21) + (95 \times .12) \times (100 \times .04)$

= 85 expected units.

If the safety stock is not provided, then shortages will occur whenever the demand during lead time exceed 85 units. The expected number of units short per lead time is computed in the following table.

Reorder level (r)	Satety stock level (w)	Lead time demand (d)	Shortages during lead time ($d - r$)	Prob. of demand during L [$p(d)$]	$(d - r)p(d)$	Expected units short per lead time $\Sigma(d - r)\, p(d)$
85	0	85	0	0.24	0	2.85
		90	5	0.21	1.05	
		95	10	0.12	1.20	
		100	15	0.04	0.50	
90	5	90	0	0.21	0	1.0
		95	5	0.12	0.6	
		100	10	0.04	0.4	
95	10	95	0	0.12	0	0.2
		100	5	0.04	0.2	
100	15	100	0	0.04	0	0

From above table it may be observed that whenever there is no safety stock, the expected number of units short per lead time is 2.85. But, when the buffer stock of 5, 10, and 15 units is provided, the expected shortage is 1.0, 0.2 and 0, respectively.

The annual carrying costs due to safety stock is 0, 50, 100 and 150 at the rate of Rs. 10 per unit for safety stock ranging from 0 to 15 units. These costs and the expected annual shortage cost is computed as given in the following table :

Safety stock level (w)	Expected shortage per lead time $\Sigma(d - r)p(d)$	Per unit shortage cost (C_2)	Reorder cycle per year (D/Q^*)	Expected annual shortage cost $C_2(D/Q^*)\Sigma(d - r)p(d)$
0	2.85	10	5	142.5
5	1.00	10	5	50.0
10	0.20	10	5	10.0
15	0	10	5	0

The carrying costs of various safety stock levels and expected annual shortage cost are combined together in the following table to show the inventory trade-off between the two.

Safety stock level	Safety stock carrying cost	Expected annual shortage cost	Total annual expected cost
0	0	142.5	142.5
5	50	50.0	100.0
10	100	10.0	110.0
15	150	0	150.0

From this table we observe that total annual expected cost reaches a minimum level at the safety stock level of 5 units. Hence, optimal reorder level, $r = w + d_L = 5 + 85 = 90$ units.

Model 9(b) : *P* system. In the *P* system the policy is to place an order at each of the reorder time points $0, t, 2t, \ldots..$ on the planning period, the order quantity being just sufficient to bring the stock to a maximum level S, say. Since demand is probabilistic in nature, the stock height at a reorder point will be a random variable so that the order quantity will also be random. Further, to reduce the chance of facing a shortage during a combined lead time (L) and reorder interval (t), a safety stock w is maintained. The safety stock is the difference between S and the sum of expected demand (d_L) during a lead time and the expected demand (d_t) during a reorder interval, i.e.,

$$w = S - (d_t + d_L)$$

$$\text{or,} \quad S = d_t + d_L + w.$$

(Here also, if L be random, d_L denotes the expected demand over average lead time.)

Let (O, T) be the planning period, and Y denote the demand during a combined reorder interval and lead time (or average lead time) period, which is a random variable with c.d.f. $F(y)$, $y \geq 0$.

Let the costs be

C_S = ordering cost per order

C_1 = carrying cost per unit quantity per unit time

C_2 = shortage cost per unit short per unit time.

The decision variables are (t, w) which are to be determined so as to minimize the total cost over (O, T). For simplicity sake, we assume t and w to be independent.

An approximation to t is obtained by applying the *EOQ* formula in Model 1. This value of t is used in the cost expression to find the optimum value of w. We may, therefore, consider only that part of total cost which is affected by w. This cost is given by

$$C(w) = \text{total carrying cost of } w \text{ over } (O, T) + \text{total shortage cost over } (O, T)$$

$$= C_1 wT + C_2 \frac{t}{T} \sum_{d_t + D_L + w}^{\infty} (y - d_t - d_L - w) dF(y),$$

since there are in all $\frac{t}{T}$ lead time + reorder interval in the planning horizon.

Here the argument is that on an average the safety stock w is carried unused over the entire planning period so that the carrying cost for this stock is $C_1 wT$.

However, in reality a part or whole of the safety stock may be consumed if $Y > d_t + d_L$. As such, the cost expression would be

$$C(w) = C_1 \frac{t}{T} \sum_{0}^{d_t + d_L + w} (d_t + d_L + w - y) dF(y) + C_2 \frac{T}{t} \sum_{d_t + D_L + w}^{\infty} (y - d_t - d_L - w) dF(y),$$

Advantages of Inventory Control

The main advantages of inventory control are :

1. Inventory control ensures an adequate supply of items to customers and avoids shortages as far as possible at the minimum cost.
2. It makes use of available capital (and /or storage space) in a most effective way and avoids an unnecessary expenditure on high inventory etc.
3. The risk of loss due to change in prices of items is reduced.
4. It ensures a smooth and efficient running of the organization.
5. It provides advantages of quality discounts on bulk purchases.
6. It serves as a buffer stock required due to delay in supply.
7. It eliminates the possibility of duplicate ordering.

8. It helps to minimize the loss due to deterioration, obsolescence, damages or pilferage, etc.

9. It helps in maintaining the economy by absorbing some of the fluctuations when the demand for an item fluctuates or is seasonal.

10. It minimizes and controls accumulation and build-up of surplus stock, and removes the dead movable surplus stock as far as possible.

11. It utilizes benefits of price fluctuations.

Thus, it may be concluded that with the help of a good inventory, a firm is able to make purchases in economic lots, to maintain continuity of operations, to avoid small time consuming orders and to guarantee prompt delivery of finished goods.

8.6 Questions

1. A particular item has an annual demand of 900 units. The carrying cost is Rs. 2 per unit per year and the ordering cost is Rs. 90 per order. Find

   (i) the economic order quantity, and (ii) the number of orders to be placed per annum.

If the purchase price per unit is Re. 0.50, what is the total cost per year?

2. A certain product has demand of 25 units per month and the items are withdrawn uniformly. Each time a production run is made the setup cost is Rs. 15. The production cost is Re. 1 per item and inventory carrying cost is Re. 0.30 per item pr month.

   (i) Assuming shortages are not allowed, determine how often to make a production run and what size it should be.

   (ii) If shortage cost is Rs. 1.50 per item short per month, determine how often to make a production run and what size it should be.

3. Consider a shop which produces and stocks three items. The management desires never to have an investment in inventory of more than Rs. 15000. The iems are produced in lots. The demand rate for each item is constant and can be assumed to be deterministic. No backorders are allowed. The pertinent data for the items are given in

the table below. The carrying cost on each item is 20% of average inventory valuation per annum. Determine the optimal lot size for each item.

Item	1	2	3
Demand rate (units per year)	1000	500	2000
Variable cost (Rs. per unit)	20	100	50
Set-up cost per lot (Rs.)	50	75	100

4. Demand for a particular item is 2000 units per year. Unit cost is Rs. 5. Carrying cost is 12 percent per year and ordering cost is Rs. 10. Find the economic order quantity. If a one percent discount is offered if the ordered quantity is between 600 and 1000, should it be taken? What will the EOQ be in this case?

5. An ice-cream company sells one of its types by weight. If the product is not sold on the day it is prepared, it can be sold at a loss of 50 paise per Kg. But there is an unlimited market for one day old ice-cream. On the other hand, the company makes a profit of Rs. 3.20 on every Kg. of ice-cream sold on the day it is prepared. Past daily orders form a distribution with $f(x) = 0.02 - 0.002x$, $0 \leq x \leq 100$. How many Kg. of ice-cream should the company prepare everyday?

6. Demand for a particular product is probabilistic and is as follows :

x :	0	1	2	3	4
$p(x)$:	0.01	0.20	0.39	0.20	0.20

The cost of producing one unit is Rs. 5000 per unit, the carrying cost per unit per year is Rs. 500 and the shorage cost per unit short per year is Rs. 1500. Find the optimal production level.

8.7 References

Ackoff, R. L. and M. W. Sasieni (1968). *Fundamentals of Operational Research,* Wiley, New York.

Beneka, R. and R. Ninterboev (1973). *Linear Programming : Applications to Agriculture,* lowa State University Press.

Budnick, F. S., R. Mojena and T. E. Vollmann (1977). *Principles of Operations Research for Management,* Rechard D. Irwin Inc.

Churchman, C. W., R. Ackoff and E. L. Arnoff (1957). *Introduction to Operations Research,* Wiley, New York.

Dantzing, G. B. (1963). *Linear Programming and Extensions, Princeton University Press,* New Jersey.

Ecker, J. G. and M. Kupferschmid (1988). *Introduction to Operations Research,* John Wiley & Sons.

Gass, S. I. (1969). *Linear Programming, 3/e, McGraw-Hill Kogakusha,* Tokyo.

Ghosal, A. (1969). *Elements of Operations Research,* Hindustan Publishing Corporation, Delhi.

Gross, D. and C.M. Harsis (1985). *Fundamentals of Queueing Theory.* John Wiley & Sons, New York.

Hadley, G. (1962). *Linear Programming,* Addison Wesley Reading Mass Massachusetts.

Hadley, G. and T.M. Whitin (1963). *Analysis of Inventory Systems,* Prentice Hall, Englewood Cliffs, N.J.

Kleinrock, L. (1975). *Queueing Systems, Vol. 1 : Theory,* John Wiley & Sons, New York.

McKinsey, J. (1952). *Introduction to the Theory of Games,* Mc-Graw Hill, New York.

Miller, D. W. and M.K. Starr (1973). *Executive Decisions and Operations Research,* Prentice-Hall, 2/e Engloewood Cliffs, N. J.

Morse, P.M. (1958). *Queues, Inventories and Maintenance,* Wiley, New York.

Naddor, E. (1966). *Inventory System.* Wiley, New York.

Prabhu, N. U. (1965). *Queues and Inventories,* Wiley, New York.

Rao, S. S. (1978). *Optimization—Theory and Applications,* Wiley Eastern, New Delhi.

Rao, K. V. (1986). *Management Science,* McGraw-Hill Book Co., Singapore.

Saaty, T. L. (1961). *Elements of Queueing Theory,* McGraw-Hill, New York.

Sasieni, M., A. Yaspan and L. Firedman (1959). *Operations Research (Methods and Problems)*, Wiley, New York.

Starr, M. K. and D. W. Miller (1974). *Inventory Control—Theory and Practice,* 2nd Edition, Prentice-Hall, Englewood Cliffs, N.J.

Taha, H.A. (1995). *Operations Research—An Introduction*, Prentice Hall of India, New Delhi.

Wagner, H.M. (1975). *Principles of Operations Research*, Prentice-Hall of India, New Delhi.